Advanced Linear Algebra – Week 1

Vector Spaces

This week we will learn about:

- Abstract vector spaces,
- How to do linear algebra over fields other than $\mathbb{R}$,
- How to do linear algebra with things that don't look like vectors, and
- Linear combinations and linear (in)dependence (again).

Extra reading and watching:

- Sections 1.1.1 and 1.1.2 in the textbook
- Lecture videos 1, 1.5, 2, 3, and 4 on YouTube
- Vector space at Wikipedia
- Complex number at Wikipedia
- Linear independence at Wikipedia

Extra textbook problems:

$\star$ 1.1.1, 1.1.4(a–f,h)

$\star\star$ 1.1.2, 1.1.5–1.1.7, 1.1.9, 1.1.11, 1.1.18, 1.1.19

$\star\star\star$ 1.1.10, 1.1.13, 1.1.20, 1.1.22, 1.1.23

☠ none this week

In the previous linear algebra course (MATH 2221), for the most part you learned how to perform computations with vectors and matrices. Some things that you learned how to compute include:

In this course, we will be working with many of these same objects, but we are going to generalize them and look at them in strange settings where we didn't know we could use them. For example:

In order to use our linear algebra tools in a more general setting, we need a proper definition that tells us what types of objects we can consider. The following definition makes this precise, and the intuition behind it is that the objects we work with should be "like" vectors in $\mathbb{R}^n$:

Definition 1.1 — Vector Space

Let $\mathcal{V}$ be a set and let $\mathbb{F}$ be a field. Let $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $c \in \mathbb{F}$, and suppose we have defined two operations called *addition* and *scalar multiplication* on $\mathcal{V}$. We write the addition of $\mathbf{v}$ and $\mathbf{w}$ as $\mathbf{v} + \mathbf{w}$, and the scalar multiplication of c and $\mathbf{v}$ as $c\mathbf{v}$.

If the following ten conditions hold for all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathcal{V}$ and all $c, d \in \mathbb{F}$, then $\mathcal{V}$ is called a **vector space** and its elements are called **vectors**:

a) $\mathbf{v} + \mathbf{w} \in \mathcal{V}$ (closure under addition)

b) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ (commutativity)

c) $(\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$ (associativity)

d) There exists a "zero vector" $\mathbf{0} \in \mathcal{V}$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.

e) There exists a vector $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

f) $c\mathbf{v} \in \mathcal{V}$ (closure under scalar multiplication)

g) $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ (distributivity)

h) $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ (distributivity)

i) $c(d\mathbf{v}) = (cd)\mathbf{v}$

j) $1\mathbf{v} = \mathbf{v}$

Some points of interest are in order:

- A field $\mathbb{F}$ is basically just a set on which we can add, subtract, multiply, and divide according to the usual laws of arithmetic.

- Vectors might not look at all like what you're used to vectors looking like. Similarly, vector addition and scalar multiplication might look weird too (we will look at some examples).

Example. $\mathbb{R}^n$ *is a vector space.*

Example. $\mathcal{F}$, *the set of all functions* $f : \mathbb{R} \to \mathbb{R}$, *is a vector space.*

Example. *$\mathcal{M}_{m,n}(\mathbb{F})$, the set of all $m \times n$ matrices with entries from $\mathbb{F}$, is a vector space.*

Be careful: the operations that we call vector addition and scalar multiplication just have to satisfy the 10 axioms that were provided—they do not have to look *anything* like what we usually call "addition" or "multiplication."

Example. *Let $\mathcal{V} = \{x \in \mathbb{R} : x > 0\}$ be the set of positive real numbers. Define addition $\oplus$ on this set via usual multiplication of real numbers (i.e., $\mathbf{x} \oplus \mathbf{y} = xy$), and scalar multiplication $\odot$ on this set via exponentiation (i.e., $c \odot \mathbf{x} = x^c$). Show that this is a vector space.*

OK, so vectors and vector spaces can in fact look quite different from $\mathbb{R}^n$. However, doing math with them isn't much different at all: almost all facts that we proved in MATH 2221 actually only relied on the ten vector space properties provided a couple pages ago.

Thus we will see that really not much changes when we do linear algebra in this more general setting. We will re-introduce the core concepts again (e.g., subspaces and linear independence), but only very quickly, as they do not change significantly.

Complex Numbers

As mentioned earlier, the field $\mathbb{F}$ we will be working with throughout this course will always be $\mathbb{R}$ (the real numbers) or $\mathbb{C}$ (the complex numbers). Since complex numbers make linear algebra work so nicely, we give them a one-page introduction:

- We define i to be a number that satisfies $i^2 = -1$ (clearly, i is not a member of $\mathbb{R}$).
- An **imaginary number** is a number of the form bi, where $b \in \mathbb{R}$.
- A **complex number** is a number of the form $a + bi$, where $a, b \in \mathbb{R}$.
- Arithmetic with complex numbers works how you might naively expect:

$$(a + bi) + (c + di) =$$

$$(a + bi)(c + di) =$$

- Much like we think of $\mathbb{R}$ as a line, we can think of $\mathbb{C}$ as a plane, and the number $a + bi$ has coordinates (a, b) on that plane.
- The **length** (or **magnitude**) of the complex number $a + bi$ is $|a + bi| = \sqrt{a^2 + b^2}$.
- The **complex conjugate** of the complex number $a + bi$ is $\overline{a + bi} = a - bi$.
- We can use the previous facts to check that $(a + bi)\overline{(a + bi)} = |a + bi|^2$.
- We can also divide by (non-zero) complex numbers:

$$\frac{a + bi}{c + di} =$$

Subspaces

It will often be useful for us to deal with vector spaces that are contained within other vector spaces. This situation comes up often enough that it gets its own name:

Definition 1.2 — Subspace

If $\mathcal{V}$ is a vector space and $\mathcal{S} \subseteq \mathcal{V}$, then $\mathcal{S}$ is a **subspace** of $\mathcal{V}$ if $\mathcal{S}$ is itself a vector space with the same addition and scalar multiplication as $\mathcal{V}$.

It turns out that checking whether or not something is a subspace is much simpler than checking whether or not it is a vector space. In particular, instead of checking all ten vector space axioms, you only have to check two:

Theorem 1.1 — Determining if a Set is a Subspace

Let $\mathcal{V}$ be a vector space and let $\mathcal{S} \subseteq \mathcal{V}$ be non-empty. Then $\mathcal{S}$ is a subspace of $\mathcal{V}$ if and only if the following two conditions hold for all $\mathbf{v}, \mathbf{w} \in \mathcal{S}$ and all $c \in \mathbb{F}$:

a) $\mathbf{v} + \mathbf{w} \in \mathcal{S}$ (closure under addition)

b) $c\mathbf{v} \in \mathcal{S}$ (closure under scalar multiplication)

Proof. For the "only if" direction,

For the "if" direction,

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Example. *Is $\mathcal{P}^p$, the set of real-valued polynomials of degree at most p, a subspace of $\mathcal{F}$?*

Example. *Is the set of $n \times n$ real symmetric matrices a subspace of $\mathcal{M}_n(\mathbb{R})$?*

Example. *Is the set of 2×2 matrices with determinant 0 a subspace of $\mathcal{M}_2$?*

Spans, Linear Combinations, and Independence

We now present some definitions that you likely saw (restricted to $\mathbb{R}^n$) in your first linear algebra course. All of the theorems and proofs involving these definitions carry over just fine when replacing $\mathbb{R}^n$ by a general vector space $\mathcal{V}$.

Definition 1.3 — Linear Combinations

Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}$, let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathcal{V}$, and let $c_1, c_2, \ldots, c_k \in \mathbb{F}$. Then every vector of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

Example. *Is a linear combination of and ?*

Example. *Is a linear combination of and ?*

Definition 1.4 — Span

Let $\mathcal{V}$ be a vector space and let $B \subseteq \mathcal{V}$ be a set of vectors. Then the **span** of B, denoted by $\operatorname{span}(B)$, is the set of all (finite!) linear combinations of vectors from B:

$$\operatorname{span}(B) \stackrel{\text{def}}{=} \left\{ \sum_{j=1}^{k} c_j \mathbf{v}_j \;\middle|\; k \in \mathbb{N},\ c_j \in \mathbb{F} \text{ and } \mathbf{v}_j \in B \text{ for all } 1 \leq j \leq k \right\}.$$

Furthermore, if $\operatorname{span}(B) = \mathcal{V}$ then $\mathcal{V}$ is said to be **spanned** by B.

Example. *Show that the polynomials* $1, x,$ *and* x^2 *span* $\mathcal{P}^2$.

Example. *Is* e^x *in the span of* $\{1, x, x^2, x^3, \ldots\}$*?*

Example. *Let $E_{i,j}$ be the matrix with a 1 in its (i,j)-entry and zeros elsewhere. Show that $\mathcal{M}_2$ is spanned by $E_{1,1}, E_{1,2}, E_{2,1}$, and $E_{2,2}$.*

Example. *Determine whether or not the polynomial $r(x) = x^2 - 3x - 4$ is in the span of the polynomials $p(x) = x^2 - x + 2$ and $q(x) = 2x^2 - 3x + 1$.*

Our primary reason for being interested in spans is that the span of a set of vectors is always a subspace (and in fact, we will see shortly that every subspace can be written as the span of some vectors).

Theorem 1.2 — Spans are Subspaces

Let $\mathcal{V}$ be a vector space and let $B \subseteq \mathcal{V}$. Then $\operatorname{span}(B)$ is a subspace of $\mathcal{V}$.

Proof. We just verify that the two defining properties of subspaces are satisfied:

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Definition 1.5 — Linear Dependence and Independence

Let $\mathcal{V}$ be a vector space and let $B \subseteq \mathcal{V}$ be a set of vectors. Then B is **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{F}$, at least one of which is not zero, and vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in B$ such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If B is not linearly dependent then it is called **linearly independent**.

There are a couple of different ways of looking at linear dependence and independence. For example:

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent if and only if

  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ implies

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent if and only if there exists a particular j such that

  $\mathbf{v}_j$ is a

In particular, a set of two vectors is linearly dependent if and only if they are scalar multiples of each other.

Example. *Is the set of polynomials* $\Big\{ \qquad\qquad , \qquad\qquad \Big\}$ *linearly dependent or independent?*

Example. *Is the set of matrices* $\Big\{ \qquad\qquad , \qquad\qquad , \qquad\qquad \Big\}$ *linearly dependent or independent?*

Example. *Is the set of functions* $\{\sin^2(x), \cos^2(x), \cos(2x)\} \subset \mathcal{F}$ *linearly dependent or independent?*

Roughly, the reason that this final example didn't devolve into something we can just compute via "plug and chug" is that we don't have a nice basis for $\mathcal{F}$ that we can work with. This contrasts with the previous two examples (polynomials and matrices), where we do have nice bases, and we've been working with those nice bases already (perhaps without even realizing it).

We will talk about bases in depth next week!