Bases and Coordinate Systems

This week we will learn about:

- Bases of vector spaces,
- How to change bases in vector spaces, and
- Coordinate systems for representing vectors.

Extra reading and watching:

- Sections 1.1.3–1.2.2 in the textbook
- Lecture videos 5, 6, 7, and 8 on YouTube
- Basis (linear algebra) at Wikipedia
- Change of basis at Wikipedia

Extra textbook problems:

⋆ 1.1.3, 1.1.4(g), 1.2.1, 1.2.4(a–c,f,g)

⋆⋆ 1.1.15, 1.1.16, 1.2.2, 1.2.5, 1.2.7, 1.2.29

⋆⋆⋆ 1.1.17, 1.1.21, 1.2.9, 1.2.23

☠ 1.2.34

In introductory linear algebra, we learned a bit about bases, but we weren't really able to do too much with them when we were restricted to $\mathbb{R}^n$. Now that we are dealing with general vector spaces, bases will really start to shine, as they let us turn almost any vector space calculation into a familiar calculation in $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Definition 2.1 — Bases

A **basis** of a vector space $\mathcal{V}$ is a set of vectors in $\mathcal{V}$ that

a) spans $\mathcal{V}$, and

b) is linearly independent.

Be careful: A vector space can have many bases that look very different from each other!

Example. *Let* $\mathbf{e}_j$ *be the vector in* $\mathbb{R}^n$ *with a* 1 *in its* j*-th entry and zeros elsewhere. Show that* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ *is a basis of* $\mathbb{R}^n$*.*

*[Side note: This is called the **standard basis** of* $\mathbb{R}^n$*.]*

Example. *Let* $E_{i,j} \in \mathcal{M}_{m,n}$ *be the matrix with a* 1 *in its* (i, j)*-entry and zeros elsewhere. Show that* $\{E_{1,1}, E_{1,2}, \ldots, E_{m,n}\}$ *is a basis of* $\mathcal{M}_{m,n}$*.*

*[Side note: This is called the **standard basis** of* $\mathcal{M}_{m,n}$*.]*

Example. *Show that the set of polynomials* $\{1, x, x^2, \ldots, x^p\}$ *is a basis of* $\mathcal{P}^p$.
[Side note: This is called the ***standard basis*** *of* $\mathcal{P}^p$.*]*

Example. *Is* $\{1 + x, 1 + x^2, x + x^2\}$ *a basis of* $\mathcal{P}^2$*?*

In the previous example, to answer a linear algebra question about $\mathcal{P}^2$, we converted the question into one about matrices, and then we answered that question instead. *This works in complete generality!* We will now start using bases to see that almost any linear algebra question that I can ask you about any vector space can be rephrased in terms of more "concrete" things like vectors in $\mathbb{R}^n$ and matrices in $\mathcal{M}_{m,n}$.

Our starting point is the following theorem:

Theorem 2.1 — Uniqueness of Linear Combinations

Let $\mathcal{V}$ be a vector space and let B be a basis for $\mathcal{V}$. Then for every $\mathbf{v} \in \mathcal{V}$, there is exactly one way to write $\mathbf{v}$ as a linear combination of the basis vectors in B.

Proof. The proof is very similar to the corresponding statement about bases of $\mathbb{R}^n$ from the previous course:

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The above theorem tells us that the following definition makes sense:

Definition 2.2 — Coordinate Vectors

Suppose $\mathcal{V}$ is a vector space over a field $\mathbb{F}$ with a finite (ordered) basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, and $\mathbf{v} \in \mathcal{V}$. Then the unique scalars c_1, c_2, …, $c_n \in \mathbb{F}$ for which

are called the **coordinates** of $\mathbf{v}$ with respect to B, and the vector

is called the **coordinate vector** of $\mathbf{v}$ with respect to B.

The above theorem and definition tell us that if we have a basis of a vector space, then we can treat the vectors in that space just like vectors in $\mathbb{F}^n$ (where n is the number of vectors in the basis). In particular, coordinate vectors respect vector addition and scalar multiplication "how you would expect them to:"

Example. *Find the coordinate vector of... ...with respect to the basis* $\{1, x, x^2\}$ *of* $\mathcal{P}^2$.

More generally,

Be careful: The order in which the basis vectors appear in B affects the order of the entries in the coordinate vector. This is kind of janky (technically, sets don't care about order), but everyone just sort of accepts it.

Example. *Find the coordinate vector of... ...with respect to the basis* $\{x^2, x, 1\}$ *of* $\mathcal{P}^2$.

Example. *Find the coordinate vector of... ...with respect to the basis* $\{1 + x, 1 + x^2, x + x^2\}$ *of* $\mathcal{P}^2$.

Notice that when we change the basis B that we are working with, coordinate vectors $[\mathbf{v}]_B$ change as well (even though $\mathbf{v}$ itself does not change). We will soon learn how to easily change coordinate vectors from one basis to another, but first we need to know that all coordinate vectors have the same number of entries:

Theorem 2.2 — Linearly Independent Sets versus Spanning Sets

Let $\mathcal{V}$ be a vector space with a basis B of size n. Then

a) Any set of more than n vectors in $\mathcal{V}$ must be linearly dependent, and

b) Any set of fewer than n vectors cannot span $\mathcal{V}$.

Proof. For (a), suppose there are $m > n$ vectors, which we call $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. We want to solve $c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$. This is the same as

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The previous theorem immediately implies the following one (which we proved for subspaces of $\mathbb{R}^n$ in the previous course):

Corollary 2.3 — Uniqueness of Size of Bases

Let $\mathcal{V}$ be a vector space that has a basis consisting of n vectors. Then *every* basis of $\mathcal{V}$ has exactly n vectors.

Based on the previous corollary, the following definition makes sense:

Definition 2.3 — Dimension of a Vector Space

A vector space $\mathcal{V}$ is called...

a) **finite-dimensional** if it has a finite basis, and its **dimension**, denoted by $\dim(\mathcal{V})$, is the number of vectors in one of its bases.

b) **infinite-dimensional** if it has no finite basis, and we say that $\dim(\mathcal{V}) = \infty$.

Example. *Let's compute the dimension of some vector spaces that we've been working with.*

Before proceeding, it is worth noting that every finite-dimensional vector space has a basis. The situation for infinite-dimensional vector spaces, however, is a bit murky...

Change of Basis

Sometimes one basis (i.e., coordinate system) will be much easier to work with than another. While it is true that the standard basis (of $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathcal{P}^p$, or $\mathcal{M}_{m,n}$) is often the simplest one to use for calculations, other bases often reveal hidden structure that can make our lives easier.

We will discuss how to find these other bases shortly, but for now let's talk about how to convert coordinate systems from one basis to another.

Definition 2.4 — Change-of-Basis Matrix

Suppose $\mathcal{V}$ is a vector space with bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and C. The **change-of-basis matrix** from B to C, denoted by $P_{C\leftarrow B}$, is the $n \times n$ matrix whose columns are the coordinate vectors $[\mathbf{v}_1]_C, [\mathbf{v}_2]_C, \ldots, [\mathbf{v}_n]_C$:

The following theorem shows that the change-of-basis matrix $P_{C\leftarrow B}$ does exactly what its name suggests: it converts coordinate vectors from basis B to basis C.

Theorem 2.4 — Change-of-Basis Matrices

Suppose B and C are bases of a finite-dimensional vector space $\mathcal{V}$, and let $P_{C\leftarrow B}$ be the change-of-basis matrix from B to C. Then

a) $P_{C\leftarrow B}[\mathbf{v}]_B = [\mathbf{v}]_C$ for all $\mathbf{v} \in \mathcal{V}$, and

b) $P_{C\leftarrow B}$ is invertible and $P_{C\leftarrow B}^{-1} = P_{B\leftarrow C}$.

Furthermore, $P_{C\leftarrow B}$ is the unique matrix with property (a).

Some notes are in order:

Proof of Theorem 2.4. For (a), suppose $\mathbf{v} \in \mathcal{V}$ and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, so that $[\mathbf{v}]_B = (c_1, c_2, \ldots, c_n)$. Then

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Example. *Find the change-of-basis matrix* $P_{C\leftarrow B}$ *for the bases* $B = \{1, x, x^2\}$ *and* $C = \{1 + x, 1 + x^2, x + x^2\}$ *of* $\mathcal{P}^2$. *Then find the coordinate vector of...*
...with respect to C.

The previous example was not too difficult, since B happened to be the standard basis of $\mathcal{P}^2$. However, if it *weren't* the standard basis, then computing the columns of $P_{C\leftarrow B}$ would have been much more difficult (each column would require us to solve a linear system). The following theorem gives a better way of computing $P_{C\leftarrow B}$ in general:

Theorem 2.5 — Computing Change-of-Basis Matrices

Let $\mathcal{V}$ be a finite-dimensional vector space with bases B, C, and E. Then the reduced row echelon form of the augmented matrix

$$\left[\ P_{E\leftarrow C} \ \middle|\ P_{E\leftarrow B}\ \right] \quad \text{is}$$

Proof. Suppose for now that we just wanted to compute $[\mathbf{v}_j]_C$ (the j-th column of $P_{C\leftarrow B}$).

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It is worth making some notes about the above theorem:

- $P_{E\leftarrow B}$ and $P_{E\leftarrow C}$ are both...

- This method for computing $P_{C\leftarrow B}$ is almost identical to the method you learned in introductory linear algebra for computing...

Example. *Find the change-of-basis matrix* $P_{C\leftarrow B}$*, where*

$$B = \Big\{ \qquad , \qquad , \qquad \Big\} \quad \textit{and} \quad C = \Big\{ \qquad , \qquad , \qquad \Big\}$$

are bases of $\qquad$. *Then compute* $[\mathbf{v}]_C$ *if* $[\mathbf{v}]_B = (1, 2, 3)$.