

LINEAR TRANSFORMATIONS

This week we will learn about:

- Linear transformations,
- The standard matrix of a linear transformation,
- Composition and powers of linear transformations, and
- Change of basis for linear transformations.

Extra reading and watching:

- Section 1.2.3 in the textbook
- Lecture videos [9](#), [10](#), [11](#), and [12](#) on YouTube
- [Linear map](#) at Wikipedia
- [Transformation matrix](#) at Wikipedia

Extra textbook problems:

- ★ 1.2.3
- ★★ 1.2.6, 1.2.11, 1.2.32
- ★★★ 1.2.12, 1.2.28, 1.2.30
- ☞ none this week

Last week, we learned that we could use bases to represent vectors in (finite-dimensional) vector spaces very concretely as tuples in \mathbb{R}^n (or \mathbb{F}^n , where \mathbb{F} is the field you're working in), thus turning almost any vector space problem into one that you learned how to solve in the previous course.

We will now introduce linear transformations between general vector spaces, and see that bases let us similarly think of any linear transformation (on finite-dimensional vector spaces) as a matrix in $\mathcal{M}_{m,n}$.

Definition 3.1 — Linear Transformations

Let \mathcal{V} and \mathcal{W} be vector spaces over the same field \mathbb{F} . A **linear transformation** is a function $T : \mathcal{V} \rightarrow \mathcal{W}$ that satisfies the following two properties:

- a)
- b)

Example. *Every matrix transformation is a linear transformation. That is,*

Example. *Is the function $T : \mathcal{M}_{m,n} \rightarrow \mathcal{M}_{n,m}$ that sends a matrix to its transpose a linear transformation?*

Example. *Is the function $\det : \mathcal{M}_n \rightarrow \mathbb{R}$ that sends a matrix to its determinant a linear transformation?*

Example. *Is the differentiation map $D : \mathcal{D} \rightarrow \mathcal{F}$, which sends a differentiable function to its derivative, a linear transformation?*

Before proceeding to prove things about linear transformations, we make some notes:

- We can sometimes consider the same linear transformation as acting on different vector spaces. For example, we can similarly consider D as a linear transformation from \mathcal{P}^3 to \mathcal{P}^2 .
- For all linear transformations $T : \mathcal{V} \rightarrow \mathcal{W}$, it is true that $T(\mathbf{0}) = \mathbf{0}$.

- The **zero transformation** $O : \mathcal{V} \rightarrow \mathcal{W}$ is the one defined by
- The **identity transformation** $I : \mathcal{V} \rightarrow \mathcal{V}$ is the one defined by

The Standard Matrix

We now do for linear transformations what we did for vectors last week: we give them “coordinates” so that we can explicitly write them down using numbers in the ground field.

Theorem 3.1 — Standard Matrix of a Linear Transformation

Let \mathcal{V} and \mathcal{W} be vector spaces with bases B and D , respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and \mathcal{W} is m -dimensional. A function $T : \mathcal{V} \rightarrow \mathcal{W}$ is a linear transformation if and only if there exists a matrix $[T]_{D \leftarrow B} \in \mathcal{M}_{m,n}$ for which

Furthermore, the unique matrix $[T]_{D \leftarrow B}$ with this property is called the **standard matrix** of T with respect to the bases B and D , and it is

Before proving this theorem, we make some notes:

- The matrix $[T]_{D \leftarrow B}$ tells us how to convert coordinate vectors of $\mathbf{v} \in \mathcal{V}$ to coordinate vectors of $T(\mathbf{v}) \in \mathcal{W}$.
- Using this theorem, we can think of every linear transformation $T : \mathcal{V} \rightarrow \mathcal{W}$ as a matrix.
- The standard matrix looks different depending on the bases B and D ,

Proof of Theorem 3.1. We just do block matrix multiplication:



Standard matrices can perhaps be made a bit simpler to understand if we draw a schematic of how they work:

Example. Find the standard matrix of the transpose map on \mathcal{M}_2 with respect to the standard basis $\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$.

Example. Find the standard matrix of the differentiation map $D : \mathcal{P}^3 \rightarrow \mathcal{P}^3$ with respect to the standard basis $\{1, x, x^2, x^3\} \subset \mathcal{P}^3$.

Composition and Powers of Linear Transformations

It is often useful to consider the effect of applying two or more linear transformations to a vector, one after another. Rather than thinking of these linear transformations as separate objects that are applied in sequence, we can combine their effect into a single new function that is called their **composition**:

The following theorem tells us that we can find the standard matrix of the composition of two linear transformations simply via matrix multiplication (as long as the bases “match up”).

Theorem 3.2 — Composition of Linear Transformations

Suppose \mathcal{V} , \mathcal{W} , and \mathcal{X} are finite-dimensional vector spaces with bases B , C , and D , respectively. If $T : \mathcal{V} \rightarrow \mathcal{W}$ and $S : \mathcal{W} \rightarrow \mathcal{X}$ are linear transformations then $S \circ T : \mathcal{V} \rightarrow \mathcal{X}$ is a linear transformation, and its standard matrix is

Proof. We just need to show that $[(S \circ T)(\mathbf{v})]_D = [S]_{D \leftarrow C} [T]_{C \leftarrow B} [\mathbf{v}]_B$ for all $\mathbf{v} \in \mathcal{V}$. To this end,

In the special case when the linear transformations that we are composing are equal to each other, we get **powers** of those transformations: ■

