Adjoints and Unitaries

This week we will learn about:

- The adjoint of a linear transformation, and
- Unitary transformations and matrices.

Extra reading and watching:

- Sections 1.4.2 and 1.4.3 in the textbook
- Lecture videos 23 and 24 on YouTube
- Unitary matrix at Wikipedia

Extra textbook problems:

⋆ 1.4.5(b,c,e,f), 1.4.8

⋆⋆ 1.4.3, 1.4.9–1.4.14, 1.4.21, 1.4.22

⋆⋆⋆ 1.4.6, 1.4.15, 1.4.18

☠ 1.4.19, 1.4.28

We now introduce the adjoint of a linear transformation, which we can think of as a way of generalizing the transpose of a real matrix to linear transformations between arbitrary inner product spaces.

Definition 6.1 — Adjoint Transformations

Suppose that $\mathcal{V}$ and $\mathcal{W}$ are inner product spaces and $T : \mathcal{V} \to \mathcal{W}$ is a linear transformation. Then a linear transformation $T^* : \mathcal{W} \to \mathcal{V}$ is called the **adjoint** of T if

For example, the adjoint of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{R})$ is

Similarly, the adjoint of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{C})$ is

So far, we have been a bit careless and referred to "the" adjoint of a matrix (linear transformation), even though it perhaps seems believable that a linear transformation might have more than one adjoint. The following theorem shows that, at least in finite dimensions, this is not actually a problem.

Theorem 6.1 — Existence and Uniqueness of Adjoints

Suppose that $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional inner product spaces. For every linear transformation $T : \mathcal{V} \to \mathcal{W}$ there exists a unique adjoint transformation $T^* : \mathcal{W} \to \mathcal{V}$. Furthermore, if B and C are orthonormal bases of $\mathcal{V}$ and $\mathcal{W}$ respectively, then

Proof. To prove uniqueness of T^*, suppose that T^* exists, let $\mathbf{v} \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{W}$, and compute $\langle T(\mathbf{v}), \mathbf{w} \rangle$ in two different ways:

■

Example. *Show that the adjoint of the transposition map $T : \mathcal{M}_{m,n} \to \mathcal{M}_{n,m}$, with the Frobenius inner product, is also the transposition map.*

The situation presented in the above example, where a linear transformation is its own adjoint, is important enough that we give it a name:

Definition 6.2 — Self-Adjoint Transformations

Suppose that $\mathcal{V}$ is an inner product space. Then a linear transformation $T : \mathcal{V} \to \mathcal{V}$ is called **self-adjoint** if $T^* = T$.

For example, a matrix in $\mathcal{M}_n(\mathbb{R})$ is self-adjoint if and only if it is...

and a matrix in $\mathcal{M}_n(\mathbb{C})$ is self-adjoint if and only if it is...

Furthermore, a linear transformation is self-adjoint if and only if its standard matrix...

Unitary Transformations and Matrices

In situations where the norm of a vector is important, it is often desirable to work with linear transformations that do not alter that norm. We now start investigating these linear transformations.

Definition 6.3 — Unitary Transformations

Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces and let $T : \mathcal{V} \to \mathcal{W}$ be a linear transformation. Then T is said to be **unitary** if

$$\|T(\mathbf{v})\| = \|\mathbf{v}\| \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

We also say that a *matrix* is unitary if it acts as a unitary linear transformation on $\mathbb{F}^n$.

Example. *Show that the matrix* $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ *is unitary.*

Fortunately, there is a much simpler method of checking whether or not a matrix (or a linear transformations) is unitary, as demonstrated by the following theorem.

Theorem 6.2 — Characterization of Unitary Matrices

Suppose $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $U \in \mathcal{M}_n(\mathbb{F})$. The following are equivalent:

a) U is unitary,

b) $U^*U = I$,

c) $UU^* = I$,

d) $(U\mathbf{v}) \cdot (U\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$,

e) The columns of U are an orthonormal basis of $\mathbb{F}^n$, and

f) The rows of U are an orthonormal basis of $\mathbb{F}^n$.

It is worth comparing these properties to corresponding properties of invertible matrices:

Proof of Theorem 6.2. We do not prove all equivalences of this theorem – for that you can see the textbook. But we will demonstrate some of them in order to give an idea of why this theorem is true.

The equivalence of (b) and (c) follows from the fact that

To see that (d) $\implies$ (b), note that if we rearrange the equation $(U\mathbf{v}) \cdot (U\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$ slightly, we get

To see that (b) implies (a), suppose $U^*U = I$. Then for all $\mathbf{v} \in \mathbb{F}^n$ we have

To see that (b) is equivalent to (e), write U in terms of its columns $U = \begin{bmatrix} \mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n \end{bmatrix}$ and then use block matrix multiplication to multiply by U^*:

The remaining implications can be proved using similar techniques. ■

Checking whether or not a matrix is unitary is now quite simple, since we just have to check whether or not $U^*U = I$. For example, if we again return to the matrix

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

from earlier:

More generally, every rotation matrix and reflection matrix is unitary, as we now demonstrate.

Example. *Show that every rotation matrix $U \in \mathcal{M}_2(\mathbb{R})$ is unitary.*

Example. *Show that every reflection matrix $U \in \mathcal{M}_n(\mathbb{R})$ is unitary.*

In fact, the previous two examples provide exactly the intuition that you should have for unitary matrices—they are the ones that rotate and/or reflect $\mathbb{F}^n$, but do not stretch, shrink, or otherwise "distort" it. They can be thought of as "rigid" linear transformations that leave the size and shape of $\mathbb{F}^n$ in tact, but possibly change its orientation.