Subspaces, Spans, and Linear Independence

This week we will learn about:

- Subspaces,
- The span of a set of vectors, and
- Linear (in)dependence.

Extra reading and watching:

- Section 2.3 in the textbook
- Lecture videos 26, 27, 28, and 29 on YouTube
- Linear subspace at Wikipedia
- Linear independence at Wikipedia

Extra textbook problems:

⋆ 2.3.1, 2.3.2, 2.3.4

⋆⋆ 2.3.3, 2.3.5, 2.3.6, 2.3.9–2.3.11, 2.3.18, 2.3.19

⋆⋆⋆ 2.3.12, 2.3.14, 2.3.16, 2.3.22

☠ 2.3.27

Subspaces

Recall that linear systems can be interpreted geometrically as asking for the point(s) of intersection of a collection of lines or planes (depending on the number of variables involved). The following definition introduces "subspaces", which can be thought of as any-dimensional analogues of lines and planes.

Definition 7.1 — Subspaces

A **subspace** of $\mathbb{R}^n$ is a non-empty set $\mathcal{S}$ of vectors in $\mathbb{R}^n$ such that:

a) If $\mathbf{v}$ and $\mathbf{w}$ are in $\mathcal{S}$ then $\mathbf{v} + \mathbf{w}$ is in $\mathcal{S}$.

b) If $\mathbf{v}$ is in $\mathcal{S}$ and c is a scalar, then $c\mathbf{v}$ is in $\mathcal{S}$.

Properties (a) and (b) above together are equivalent to requiring that $\mathcal{S}$ is closed under linear combinations:

Example. Is the set of vectors (x, y) satisfying $y = x^2$ a subspace of $\mathbb{R}^2$?

Example. Is the set of vectors (x, y, z) satisfying $x = 3y$ and $z = -2y$ a subspace of $\mathbb{R}^3$?

Example. Is the set of vectors (x, y, z) satisfying $x = 3y + 1$ and $z = -2y$ a subspace of $\mathbb{R}^3$?

In $\mathbb{R}^3$, lines and planes through the origin are subspaces (this is hopefully not difficult to see for lines, and it can be seen for planes by using the parallelogram law):

Even though we can't visualize subspaces in higher dimensions, you should keep the line/plane intuition in mind: a subspace of $\mathbb{R}^n$ looks like a copy of $\mathbb{R}^m$ (for some $m < n$) going through the origin.

Subspaces Associated with Matrices

Let's now look at some other natural examples of subspaces that appear frequently when working with matrices.

Definition 7.2 — Matrix Subspaces

Let $A \in \mathcal{M}_{m,n}$ be an $m \times n$ matrix.

a) The **range** of A is the subspace of $\mathbb{R}^m$, denoted by $\mathrm{range}(A)$, that consists of all vectors of the form $A\mathbf{x}$.

b) The **null space** of A is the subspace of $\mathbb{R}^n$, denoted by $\mathrm{null}(A)$, that consists of all solutions $\mathbf{x}$ of the linear system $A\mathbf{x} = \mathbf{0}$.

Some remarks about these matrix subspaces are in order:

- $\mathrm{null}(A)$ is a subspace. Why?

- $\mathrm{range}(A)$ is a subspace. Why?

- The term “range” is being used here in the exact same sense as in previous courses.

Example. Describe the range and null space of the 2×3 matrix

The Span of a Set of Vectors

One way to turn a set that is *not* a subspace into a subspace is to add linear combinations to it. For example, the set containing only the vector $(2, 1)$ is not a subspace of $\mathbb{R}^2$ because

To fix this problem, we could

In general, if our starting set contains more than just one vector, we might also have to add general linear combinations of those vectors (not just their scalar multiples) in order to create a subspace. This idea of enlarging a set so as to create a subspace is an important one that we now give a name and explore.

Definition 7.3 — Span

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of those vectors is called their **span**, and is denoted by $\operatorname{span}(B)$ or $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.

For example, $\operatorname{span}\big((2, 1)\big)$ is the line through the origin and the point $(2, 1)$, as we discussed earlier.

Example. Show that $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

The natural generalization of this fact holds in all dimensions:

Example. What is $\text{span}\big((1,0,3),(-1,1,-3)\big)$ – a line, a plane, or something else?

We motivated the span of a set of vectors as a way of turning that set into a subspace. We now state (but for the sake of time, do not prove) a theorem that says the span of a set of vectors is indeed always a subspace, as we would hope.

Theorem 7.1 — Spans are Subspaces

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Then $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

In fact, you can think of the span of a set of vectors as the *smallest* subspace containing those vectors.

The range of a matrix can be expressed very conveniently as the span of a set of vectors in a way that requires no calculation whatsoever:

Theorem 7.2 — Range Equals the Span of Columns

If $A \in \mathcal{M}_{m,n}$ has columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ then $\text{range}(A) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$.

This theorem follows immediately from

For example, if we return to the 2×3 matrix from earlier, we see that its range is...

We close this section by introducing a connection between the range of a matrix and invertible matrices.

Theorem 7.3 — Spanning Sets and Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) $\text{range}(A) = \mathbb{R}^n$.

c) The columns of A span $\mathbb{R}^n$.

d) The rows of A span $\mathbb{R}^n$.

Proof. The fact that properties (a) and (c) are equivalent follows from combining...

The equivalence of properties (c) and (d) follows from the fact that

Finally, the equivalence of properties (b) and (c) follows immediately from

■

The geometric interpretation of the equivalence of properties (a) and (b) in the above theorem is

Linear Dependence and Independence

Recall from earlier that a row echelon form of a matrix can have entire rows of zeros at the end of it. For example, the reduced row echelon form of

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ -1 & 1 & -2 \end{array}\right] \quad \text{is}$$

This happens when there is some linear combination of the rows of the matrix that equals the zero row, and we interpret this roughly as saying that one row the rows of the matrix (i.e., one of the equations in the associated linear system) does not "contribute anything new." In the example above, The following definition captures this idea that a redundancy among vectors or linear equations can be identified by whether or not some linear combination of them equals zero.

Definition 7.4 — Linear Dependence and Independence

A set of vectors $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$, at least one of which is not zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If a set of vectors is not linearly dependent, it is called **linearly independent**.

For example, the set of vectors $\{(2,3),(1,0),(0,1)\}$ is linearly...

On the other hand, the set of vectors $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly...

In general, to check whether or not a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent, you should set

and then try to solve for the scalars $c_1, c_2, \ldots, c_k$. If they must all equal 0, then the set is linearly independent, and otherwise it is linearly dependent.

Example. Are these vectors linearly independent?

We saw in the previous example that we can check linear (in)dependence of a set of vectors by placing those vectors as columns in a matrix and augmenting with a $\mathbf{0}$ right-hand side. This is true in general:

Theorem 7.4 — Checking Linear Dependence

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ be vectors and let A be the $m \times n$ matrix with these vectors as its columns. The following are equivalent:

a) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly dependent set.

b) The linear system $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

Some notes about linear (in)dependence are in order:

- A set of vectors is linearly dependent if and only if at least one of the vectors can be written as a linear combination of the others.

- Every set of vectors containing the zero vector is linearly...

- Geometrically, linear dependence means that...

- For a set of just 2 vectors, linear dependence means that...

Example. Is this set linearly independent?

We close this section by introducing a connection between linear independence and invertible matrices, which we unfortunately have to state without proof due to time constraints.

Theorem 7.5 — Independence and Invertible Matrices

Let $A \in \mathcal{M}_n$. The following are equivalent:

a) A is invertible.

b) The columns of A form a linearly independent set.

c) The rows of A form a linearly independent set.