Determinants

This week we will learn about:

- Determinants of matrices, and
- That's it. Determinants, determinants, determinants.

Extra reading and watching:

- Section 3.2 in the textbook
- Lecture videos 34, 35, and 36 on YouTube
- Determinant at Wikipedia

Extra textbook problems:

⋆ 3.2.1, 3.2.3, 3.2.4, 3.2.9

⋆⋆ 3.2.5–3.2.8, 3.2.10, 3.2.12, 3.2.17

⋆⋆⋆ 3.2.14, 3.2.16, 3.2.18

☠ 3.2.19–3.2.21

We now introduce one of the most important properties of a matrix: its **determinant**, which roughly is a measure of how "large" the matrix is. More specifically, recall that...

The determinant of A, which we denote by $\det(A)$, is the area (or volume) of this image of the unit hypercube. In other words, it measures how much A expands space when acting as a linear transformation.

Let's now start looking at some of the properties of determinants, so that we can (eventually!) learn how to compute it.

Definition and Basic Properties

Before we even properly define the determinant, let's think about some properties that it should have. The first important property is that, since the identity matrix does not stretch or shrink $\mathbb{R}^n$ at all...

Next, since every $A \in \mathcal{M}_n$ expands space by a factor of $\det(A)$, and similarly each $B \in \mathcal{M}_n$ expands space by a factor of $\det(B)$...

We will also need one more property of determinants, which is a bit more difficult to see. What happens to $\det(A)$ if we multiply one of the columns of A by a scalar $c \in \mathbb{R}$?

Similarly, if we add a vector to one of the columns of a matrix, then...

In other words, the determinant is linear in the columns of a matrix (sometimes called **multilinearity**). We now *define* the determinant to be the function that satisfies this multilinearity property, as well as the other two properties that we demonstrated earlier:

Definition 9.1 — Determinant

The **determinant** is the (unique!) function $\det : \mathcal{M}_n \to \mathbb{R}$ that satisfies the following three properties:

a) $\det(I) = 1$,

b) $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathcal{M}_n$, and

c) for all $c \in \mathbb{R}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{R}^n$, it is the case that

$$\det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{v} + c\mathbf{w} \mid \cdots \mid \mathbf{a}_n\ ]\big) = \det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{v} \mid \cdots \mid \mathbf{a}_n\ ]\big) + c \cdot \det\big([\ \mathbf{a}_1 \mid \cdots \mid \mathbf{w} \mid \cdots \mid \mathbf{a}_n\ ]\big).$$

Example. Compute the determinant of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Let's start looking at some of the basic properties of the determinant. First, if $A \in \mathcal{M}_n$ is invertible then properties (a) and (b) tell us that

This makes sense geometrically, since if A expands space by a factor of $\det(A)$ then

On the other hand, if A is not invertible, then

We summarize our observations about the determinant of invertible and non-invertible matrices in the following theorem:

Theorem 9.1 — Determinants and Invertibility

Suppose $A \in \mathcal{M}_n$. Then A is invertible if and only if $\det(A) \neq 0$, and if it is invertible then $\det(A^{-1}) = 1/\det(A)$.

There are also a few other basic properties of determinants that are useful to know, so we state them here (but for time reasons we do not explicitly prove them):

Theorem 9.2 — Other Properties of the Determinant

Suppose $A \in \mathcal{M}_n$ and $c \in \mathbb{R}$. Then

a) $\det(cA) = c^n \det(A)$, and

b) $\det(A^T) = \det(A)$.

Example. Suppose $A, B \in \mathcal{M}_3$ are matrices with $\det(A) = 2$ and $\det(B) = 5$. Compute...

Computation

In order to come up with a general method of computing the determinant, we start by computing it on elementary matrices.

The elementary matrix corresponding to the row operation cR_i has the form

This matrix has determinant equal to...

The elementary matrix corresponding to the row operation $R_i + cR_j$ has the form

This matrix has determinant equal to...

The elementary matrix corresponding to the row operation $R_i \leftrightarrow R_j$ has the form

This matrix has determinant equal to...

Wait, so the determinant of a matrix can be ***negative***? But it measures area/volume!

Since multiplication on the left by an elementary matrix corresponds to performing a row operation, we can rephrase our above calculations as the following theorem:

Theorem 9.3 — Computing Determinants via Row Operations

Suppose $A, B \in \mathcal{M}_n$. If B is obtained from A via a single row operation, then their determinants are related as follows:

cR_i**:** $\det(B) = c \cdot \det(A)$,

$R_i + cR_j$**:** $\det(B) = \det(A)$, and

$R_i \leftrightarrow R_j$**:** $\det(B) = -\det(A)$.

The above theorem gives us everything we need to know to be able to compute determinants in general – row reduce A to I, keeping track of the row operations that we performed along the way, and use the fact that $\det(I) = 1$. If we cannot row reduce to I, then A is not invertible, so $\det(A) = 0$.

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

In the previous example, the determinant of the row echelon form ended up being the product of its diagonal entries. We now state this observation as a theorem:

Theorem 9.4 — Determinant of a Triangular Matrix

Let $A \in \mathcal{M}_n$ be a triangular matrix. Then $\det(A)$ is the product of its diagonal entries:

$$\det(A) = a_{1,1}a_{2,2}\cdots a_{n,n}.$$

Proof. The idea is that a triangular matrix can be row-reduced to I just by operations of the form $R_i + cR_j$ (which do not affect the determinant) and $(1/a_{1,1})R_1$, …, $(1/a_{n,n})R_n$:

∎

By using this fact, we can compute determinants a bit more quickly, by just row-reducing to row echelon form (instead of *reduced* row echelon form). This method is best illustrated with another example.

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}$.

Explicit Formulas and Cofactor Expansions

Remarkably, the determinant can be computed via an explicit formula just in terms of multiplication and addition of the entries of the matrix. Before presenting the general formula for $n \times n$ matrices, let's start with what it looks like for 2×2 matrices.

Theorem 9.5 — Determinant of 2×2 Matrices

The determinant of a 2×2 matrix is given by

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

Proof. We prove this theorem by making use of multilinearity (i.e., defining property (c) of the determinant):

Well,

Adding these two quantities together gives the desired formula. ■

The above theorem is perhaps best remembered in terms of diagonals of the matrix – the determinant of a 2×2 matrix is the product of its forward diagonal minus the product of its backward diagonal.

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

The formula for the determinant of a 3×3 matrix is somewhat more complicated:

Theorem 9.6 — Determinant of 3×3 Matrices

The determinant of a 3×3 matrix is given by

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = aei + bfg + cdh - afh - bdi - ceg.$$

Proof. Again, we make use of multilinearity (i.e., defining property (c) of the determinant) to write

Let's compute the first of the three determinants on the right by using a similar trick on its second column:

Well, these two determinants are

The computation of the remaining terms in the determinant is similar. ∎

We can also think of the formula for determinants of 3×3 matrices in terms of diagonals of the matrix – it is the sum of the products of its forward diagonals minus the sum of the products of its backward diagonals, with the understanding that the diagonals "loop around" the matrix:

Example. Compute the determinant of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

The following theorem tells us how to come up with these formulas in general, and it is just a direct generalization of the 2×2 and 3×3 formulas that we already saw.

Theorem 9.7 — Cofactor Expansion

Let $A \in \mathcal{M}_n$. For each $1 \leq i, j \leq n$, define $c_{i,j} = (-1)^{i+j} \det(\overline{A_{i,j}})$, where $\overline{A_{i,j}}$ is the matrix obtained by removing the i-th row and j-th column of A. Then the determinant of A can be computed via

$$\det(A) = a_{i,1}c_{i,1} + a_{i,2}c_{i,2} + \cdots + a_{i,n}c_{i,n} \quad \text{for all } 1 \leq i \leq n, \quad \text{and}$$
$$\det(A) = a_{1,j}c_{1,j} + a_{2,j}c_{2,j} + \cdots + a_{n,j}c_{n,j} \quad \text{for all } 1 \leq j \leq n.$$

That theorem is a mouthful! Several remarks are in order:

- If we use this theorem to compute the determinant of a 2×2 or 3×3 matrix,

- The above method of computing the determinant is called a "cofactor expansion," since the number $c_{i,j}$ is called the "(i,j)-cofactor of A."

- The theorem gives *multiple* formulas for $\det(A)$:

Example. Compute the determinant of $A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ -1 & 3 & 0 & 2 \end{bmatrix}$.

Example. Compute the determinant of $A = \begin{bmatrix} 0 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 \\ -2 & 1 & 1 & -1 & 0 \\ 1 & 0 & -3 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 \end{bmatrix}$.

In general, computing determinants via cofactor expansions is extremely inefficient. It's not too bad for 2×2, 3×3, or maybe 4×4 matrices. But for an $n \times n$ matrix A, a cofactor expansion contains $n!$ terms being added up, and each of those terms is the product of n entries of A. For example,

$$\begin{aligned}\det\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & \ell \\ m & n & o & p \end{bmatrix}\right) &= afkp - af\ell o - agjp + ag\ell n + ahjo - ahkn \\ &\quad - bekp + be\ell o + bgip - bg\ell m - bhio + bhkm \\ &\quad + cejp - ce\ell n - cfip + cf\ell m + chin - chjm \\ &\quad - dejo + dekn + dfio - dfkm - dgin + dgjm.\end{aligned}$$

Ugh! So for large matrices, use the Gaussian elminiation method instead. Nonetheless, cofactor expansions will be useful for us for theoretical reasons next week.