Isometries of Locally Unitarily Invariant Norms

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Unitarily Invariant Norms

A **unitarily invariant norm** $|||\cdot|||$ on $M_{n,m}$ is one that satisfies

$$|||UXV||| = |||X||| \quad \forall \text{ unitary operators } U \in M_n, V \in M_m.$$

Popular examples:

- operator norm: $\|X\| = \sup_{|v\rangle,|w\rangle} \big\{ |\langle v|X|w\rangle| \big\}$;
- trace norm $\|X\|_{tr} = \mathrm{Tr}|X|$;
- Frobenius norm $\|X\|_F = \sqrt{\mathrm{Tr}(X^*X)}$; and
- Schatten p-norm $\|X\|_p = \sqrt[p]{\mathrm{Tr}|X|^p}$.

Unitarily Invariant Norm Isometries

The following result (Sourour 1981; Li, Tsing 1990; Đoković, Li 1994) completely characterizes isometry groups for unitarily invariant norms:

Theorem

Let $\Phi : M_{n,m} \rightarrow M_{n,m}$ be a linear map and let $|||\cdot|||$ be a unitarily invariant norm that is not a multiple of the Frobenius norm. Then $|||\Phi(X)||| = |||X|||$ for all $X \in M_{n,m}$ if and only if there exist unitary matrices U, V such that

$$\Phi(X) = UXV \qquad \text{or} \qquad n = m \text{ and } \Phi(X) = UX^TV.$$

Our First Locally Unitarily Invariant Norm

In quantum information theory, we often consider norms like the following one on $M_m \otimes M_n$:

$$\big\|X\big\|_{S(1)} = \sup_{|v\rangle,|w\rangle,|x\rangle,|y\rangle} \Big\{ \big|(\langle v| \otimes \langle w|)X(|x\rangle \otimes |y\rangle)\big| \Big\}.$$

Properties of this norm:

- $\| \cdot \|_{S(1)}$ is not unitarily invariant;
- $\| \cdot \|_{S(1)}$ is "locally" unitarily invariant:

$$\big\|X\big\|_{S(1)} = \big\|(U_1 \otimes U_2)X(V_1 \otimes V_2)\big\|_{S(1)}.$$

Locally Unitarily Invariant Norms

A **locally unitarily invariant norm** $|||\cdot|||$ on $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_p}$ or $M_{n_1,m_1} \otimes M_{n_2,m_2} \otimes \cdots \otimes M_{n_p,m_p}$ is one that satisfies, for all unitary operators $U_i \in M_{n_i}$ and $V_i \in M_{m_i}$,

$$\big|\big|\big| |v\rangle \big|\big|\big| = \big|\big|\big| (U_1 \otimes U_2 \otimes \cdots \otimes U_p)|v\rangle \big|\big|\big|$$

or

$$|||X||| = |||(U_1 \otimes \cdots \otimes U_p)X(V_1 \otimes \cdots \otimes V_p)|||,$$

respectively.

Locally Unitarily Invariant Norms

When asking general questions about locally unitarily invariant norms, it is enough to consider them on $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_p}$.

- This is because M_{n_i,m_i} is isomorphic with $\mathbb{C}^{m_i} \otimes \mathbb{C}^{n_i}$ via the map $\Gamma_i(|b\rangle\langle a|) = |a\rangle \otimes |b\rangle$.
- The norm $|||\cdot|||$ on $M_{n_1,m_1} \otimes M_{n_2,m_2} \otimes \cdots \otimes M_{n_p,m_p}$ is locally unitarily invariant if and only if the norm $\left|\left|\left|(\Gamma_1^{-1} \otimes \cdots \otimes \Gamma_p^{-1})(\cdot)\right|\right|\right|$ on $(\mathbb{C}^{m_1} \otimes \mathbb{C}^{n_1}) \otimes \cdots \otimes (\mathbb{C}^{m_p} \otimes \mathbb{C}^{n_p})$ is locally unitarily invariant.

Locally Unitarily Invariant Norms

- Any unitarily invariant norm on $M_{n,m}$ corresponds to a locally unitarily invariant norm on $\mathbb{C}^m \otimes \mathbb{C}^n$.
- We might hope that properties that hold for unitarily invariant norms have natural analogues for locally unitarily invariant norms.
- The key difference is that we don't have singular values to help us out in the multipartite (i.e., $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$ with $p \geq 3$) case.

Locally Unitarily Invariant Norms

For example, it is well-known that a norm $|||\cdot|||$ on $M_{n,m}$ is unitarily invariant if and only if $|||AXB||| \leq \|A\| \, |||X||| \, \|B\|$ for all A, X, B. The natural generalization is:

Proposition

Let $|||\cdot|||$ be a norm on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$. Then $|||\cdot|||$ is locally unitarily invariant if and only if

$$|||(A_1 \otimes \cdots \otimes A_p)|v\rangle||| \leq \left(\prod_{i=1}^{p} \|A_i\| \right) |||\,|v\rangle||| \quad \forall A_i.$$

The Conjecture

What about isometry groups of locally unitarily invariant norms?

- In the $p = 1$ case (i.e., a unitarily invariant norm on $\mathbb{C}^n$), there is only one possible isometry group: the group of unitary operators.
- In the $p = 2$ case (i.e., a locally unitarily invariant norm on $\mathbb{C}^m \otimes \mathbb{C}^n$), the result is given by the corresponding result for unitarily invariant norms on $M_{n,m}$. The possible isometry groups are:
  - the full unitary group;
  - (if $m \neq n$) the group of unitaries of the form $U \otimes V$; or
  - (if $m = n$) the group of unitaries of the form $U \otimes V$ or $S(U \otimes V)$, where S is the "swap" operator defined by $S(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle$.

The Conjecture

The natural conjecture for the $p \geq 3$ case seems to be that the isometry group for any locally unitarily invariant norm is generated by:

- the local unitaries ("local" with respect to an arbitrary partition of the tensor factors); and
- a subset of the operators that permute the tensor factors of the same dimension.

The Conjecture

Conjecture

Let $|||\cdot|||$ *be a locally unitarily invariant norm on* $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$*. Then there exists a partition* $P = \{P_1, P_2, \ldots, P_r\}$ *of the set* $[p]$ *such that the isometry group* $\mathcal{G}$ *of* $|||\cdot|||$ *satisfies*

$$\mathcal{G} = \bigcup_{\sigma} \left\{ S_\sigma(U_1 \otimes U_2 \otimes \cdots \otimes U_r) : U_i \in \bigotimes_{j \in P_i} \mathbb{C}^{n_j} \quad \forall\, i \right\},$$

where the union is taken over a subset of the permutations $\sigma : [r] \to [r]$ *such that* $\prod_{j \in P_i} n_j = \prod_{j \in P_{\sigma(i)}} n_j$ *for all* i*.*

The Conjecture

How to prove the conjecture? It would be nice to simply extend the proof from the $p = 2$ case.

- Sourour's argument heavily relies on thinking of $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ as the bounded operators on a Hilbert space.
- Li and Tsing's argument makes use of singular values.
- Đoković and Li's argument uses Lie group theory, and seems like it might extend.

The Conjecture

Can also think of this question as a quantum information theory problem:

- Suppose you want to build a quantum computer, but you only know how to implement certain quantum gates.
- A common set-up is that you know how to implement all single-qudit gates, as well as a single two-qudit gate.
- In this setting, you can implement **any** quantum circuit, as long as your two-qudit gate is an "entangling gate".

The Conjecture

The previous slide says little more than the fact that the only group between the local unitary group and the full unitary group is the group of operators of the form $U \otimes V$ or $S(U \otimes V)$.

- What about the multipartite generalization?
- That is, if we have access to a single p-qudit entangling quantum gate, together with all 1-qudit gates, can we generate all quantum gates?
- Are the only groups between the local unitary group and the full unitary group the groups of the form described by the conjecture?

Entanglement Basics

A unit vector $|v\rangle$ represents a "pure" quantum state.

- If $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ and we can write $|v\rangle = |a\rangle \otimes |b\rangle$, we say that $|v\rangle$ is **separable**.
- The minimum k so that we can write $|v\rangle = \sum_{i=1}^{k} \alpha_i |a_i\rangle \otimes |b_i\rangle$ is called the **Schmidt rank** of $|v\rangle$ (denoted $SR(|v\rangle)$).
- We always have $1 \leq SR(|v\rangle) \leq \min\{m, n\}$.

Entanglement Basics

An positive semidefinite operator ρ satisfying $\mathrm{Tr}(\rho) = 1$ represents a general (i.e., pure or mixed) quantum state.

- If we can write $\rho = \sum_i p_i|v_i\rangle\langle v_i|$ with each $|v_i\rangle$ separable, we say that ρ is **separable**.
- The minimum k so that we can write $\rho = \sum_i p_i|v_i\rangle\langle v_i|$ with $SR(|v_i\rangle) \leq k$ is called the **Schmidt number** of ρ (denoted $SN(\rho)$).
- We always have $1 \leq SN(\rho) \leq \min\{m, n\}$.

The $S(k)$-Norms

Consider the following norms:

$$\big\|X\big\|_{S(k)} = \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle v|X|w\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

The duals of these norms are:

$$\big\|X\big\|_{S(k)}^{\circ} = \inf \Big\{ \sum_i |c_i| : X = \sum_i c_i |v_i\rangle\langle w_i| \Big\},$$

where the infimum is taken over all decompositions of X of the given form with $SR(|v_i\rangle), SR(|w_i\rangle) \leq k$.

The Dual $S(k)$-Norms

The norms $\|\cdot\|_{S(k)}^{\circ}$ completely characterize Schmidt number in the following sense:

Proposition

Let $\rho \in M_m \otimes M_n$ be a quantum state. Then $SN(\rho) \leq k$ if and only if $\|\rho\|_{S(k)}^{\circ} = 1$.

- In the $k = \min\{m, n\}$ case, the theorem is vacuously true.
- In the $k = 1$ case, this is called the "cross norm criterion" for separability.

The $S(k)$-Norms

Similarly, the norms $\| \cdot \|_{S(k)}$ completely characterize positivity of linear maps in the following sense:

Proposition

Let $\mathcal{D} : M_m \to M_n$ be the map defined by $\mathcal{D}(X) = \mathrm{Tr}(X)I$ and let $\Psi : M_m \to M_n$ be completely positive. Then $c\mathcal{D} - \Psi$ is k-positive if and only if $c \geq \big\| \sum_{i,j=1}^m |i\rangle\langle j| \otimes \Psi(|i\rangle\langle j|) \big\|_{S(k)}$.

- Alternatively, this means that $cI - X \in M_m \otimes M_n$ is a "k-entanglement witness" if and only if $\big\|X\big\| > c \geq \big\|X\big\|_{S(k)}$.

The Isometries of the $S(k)$-Norms

Theorem

Let $1 \leq k < n$ and $\Phi : M_n \otimes M_n \to M_n \otimes M_n$. Then $\big\|\Phi(X)\big\|_{S(k)} = \big\|X\big\|_{S(k)}$ for all $X \in M_n \otimes M_n$ if and only if Φ can be written as a composition of one or more of the following maps:

- $X \mapsto (U \otimes V)X(W \otimes Y)$*, where* $U, V, W, Y \in M_n$ *are unitary matrices;*
- $X \mapsto S_1 X S_2$*, where* $S_1, S_2 \in \{I, S\} \subset M_n \otimes M_n$ *and* S *is the swap operator;*
- *the transpose map* T*; and*
- *if* $k = 1$*, the partial transpose map* $(id_n \otimes T)$*.*

The Geometric Measure of Entanglement

A frequently-used measure of entanglement for multipartite pure states is the geometric measure of entanglement:

$$E(|v\rangle) := 1 - \sup_{|w_i\rangle} \left\{ \left| (\langle w_1| \otimes \cdots \otimes \langle w_p|)|v\rangle \right|^2 \right\}.$$

Theorem

Let $U \in M_{n_1} \otimes \cdots \otimes M_{n_p}$. Then $E(U|v\rangle) = E(|v\rangle)$ for all $|v\rangle$ if and only if there exist unitaries $U_i \in M_{n_i}$ and a swap operator $S_\sigma : |v_1\rangle \otimes \cdots \otimes |v_p\rangle \mapsto |v_{\sigma(1)}\rangle \otimes \cdots \otimes |v_{\sigma(p)}\rangle$ (with $n_i = n_{\sigma(i)}$ for all i) such that

$$U = S_\sigma(U_1 \otimes \cdots \otimes U_p).$$

Matricially Normed Spaces

Given a norm $|||\cdot|||$ on M_n, a family of **matrix norms** $\left\{ |||\cdot|||_m \right\}$ on $M_m \otimes M_n$ is one that satisfies $|||\cdot|||_1 = |||\cdot|||$ and

$$\left|\left|\left|(A \otimes I)X(B^* \otimes I)\right|\right|\right|_m \leq \|A\| |||X|||_p \|B\|$$

for all $A, B \in M_{p,m}$.

Matricially Normed Spaces

If we add the requirement that

$$|||X \oplus Y|||_{m+r} = \max\left\{|||X|||_m, |||Y|||_r\right\}$$

for all $X \in M_m \otimes M_n$ and $Y \in M_r \otimes M_n$, then this family of norms forms an **abstract operator space**.

Examples of abstract operator spaces:

- the operator norm $\|\cdot\|$ on $M_m \otimes M_n$; and
- the norm $\|\cdot\|_{S(k)}$ on $M_m \otimes M_n$.

Matricially Normed Spaces

Alternatively, we could fix $1 \leq r < \infty$ and add the requirement that

$$|||X \oplus Y|||_{m+r} = \sqrt[p]{|||X|||_m^p + |||Y|||_r^p}$$

for all $X \in M_m \otimes M_n$ and $Y \in M_r \otimes M_n$. Such a family of norms is called an **L^p-matricially normed space**.

Examples of L^p-matricially normed spaces:

- the trace norm on $M_m \otimes M_n$ $(p = 1)$;
- the norm $\|\cdot\|_{S(k)}^{\circ}$ on $M_m \otimes M_n$ $(p = 1)$; and
- the Schatten p-norm on $M_m \otimes M_n$.

Matricially Normed Spaces

Proposition

A norm $|||\cdot|||$ on $M_p \otimes M_n$ is locally unitarily invariant if and only if there exists a family of matrix norms $\{ |||\cdot|||_m \}$ such that $|||\cdot|||_1$ is unitarily invariant and $|||\cdot|||_p = |||\cdot|||$.

Superoperator Norms

We can bring operators "down a level" by thinking of them as vectors. By doing this, superoperators become operators. By bringing things "down a level" again, we can think of superoperators as vectors as well.

- i.e., $\mathcal{L}(M_{n,m}, M_{s,r})$ is isomorphic to $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^r \otimes \mathbb{C}^s$.
- A norm $|||\cdot|||$ on superoperators satisfies $|||\Phi||| = |||\Psi|||$ whenever $\Psi(X) \equiv U_1 \Phi(U_2 X U_3) U_4$ if and only if the corresponding norm on $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^r \otimes \mathbb{C}^s$ is locally unitarily invariant.
- If $|||\cdot|||$ is any unitarily-invariant norm on $M_{n,m}$, then the induced version of the norm on $\mathcal{L}(M_{n,m})$ can be thought of as a locally unitarily invariant norm.

Induced Schatten Superoperator Norms

For example, consider the **induced Schatten $q \to p$ norm**:

$$\big\|\Phi\big\|_{q\to p} := \sup_X \Big\{ \big\|\Phi(X)\big\|_p : \big\|X\big\|_q = 1 \Big\}.$$

Important special cases:

- $p = q = 1$: the **induced trace norm**;
- $p = q = \infty$: the **induced operator norm**; and
- $p = \infty, q = 1$: the **maximal output purity**.

Induced Schatten Superoperator Norms

Theorem

Let $q < 2 < p$. Then every isometry of the $\|\cdot\|_{q\to p}$ norm can be written as a composition of one or more of the following maps:

- *The map that sends Φ to the superoperator $\Psi(X) \equiv U_1\Phi(U_2XU_3)U_4$;*
- *The map $\Phi \mapsto \Phi \circ T$; and*
- *The map $\Phi \mapsto T \circ \Phi$.*

The Conjecture Again

Conjecture

Let $|||\cdot|||$ be a locally unitarily invariant norm on $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_p}$. Then there exists a partition $P = \{P_1, P_2, \ldots, P_r\}$ of the set $[p]$ such that the isometry group $\mathcal{G}$ of $|||\cdot|||$ satisfies

$$\mathcal{G} = \bigcup_{\sigma} \left\{ S_\sigma(U_1 \otimes U_2 \otimes \cdots \otimes U_r) : U_i \in \bigotimes_{j \in P_i} \mathbb{C}^{n_j} \quad \forall\, i \right\},$$

where the union is taken over a subset of the permutations $\sigma : [r] \to [r]$ such that $\prod_{j \in P_i} n_j = \prod_{j \in P_{\sigma(i)}} n_j$ for all i.