CP-Invariance and Complete Positivity in Quantum Information Theory

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Order of Events

- Operator Systems on Complex Matrices
- Positive and Completely Positive Maps
- Operator Systems $\leftrightarrow$ Right-CP-Invariant Cones
- Minimal and Maximal Operator Systems
- Other CP-Invariant Cones from Quantum Information Theory

What are Cones and Completely Positive Maps?

We will be mostly concerned with the space of $n \times n$ complex matrices M_n.

- A **cone** $C \subset M_n$ is a set of Hermitian operators that is invariant under positive scaling.
- A cone C is called **convex** if $X + Y \in C$ whenever $X, Y \in C$.

What are Cones and Completely Positive Maps?

We can also consider cones of linear maps acting on M_n (i.e., in $\mathcal{L}(M_n)$).

- A **cone** $C \subset \mathcal{L}(M_n)$ is a set of Hermiticity-preserving maps that is invariant under positive scaling.
- An **adjoint map** is a map of the form $X \mapsto A^\dagger X A$, where $A^\dagger$ denotes the adjoint operator of A. We will denote this map by Ad_A.
- The set of adjoint maps is a cone. Its convex hull is the cone of **completely positive maps**.

What is an Operator System?

An (abstract) **operator system on M_n** is a family of convex cones $C_m \subseteq M_m \otimes M_n$ (one cone for each $m \in \mathbb{N}$) that satisfy two properties:

1. $C_1 = M_n^+$, the cone of positive-semidefinite operators in M_n; and
2. for each $m_1, m_2 \in \mathbb{N}$ and $A \in M_{m_1,m_2}$ we have $(\mathrm{Ad}_A \otimes id_n)(C_{m_1}) \subseteq C_{m_2}$.

Intuitively, these restrictions say that each cone C_m is somehow "like" the cone M_n^+.

Examples of Operator Systems on M_n

Quantum information theorists are actually familiar with some operator systems on M_n already.

- The most natural operator system on M_n is the one that arises by making the association $M_m \otimes M_n \cong M_{mn}$ in the natural way and letting $C_m = M_{mn}^+$; the cone of positive semidefinite operators. Keep this "naïve" operator system in mind!
- We will denote this "naïve" operator system simply by M_n. We will denote other general operator systems on M_n by things like O_1 and O_2.

Examples of Operator Systems on M_n

- An operator $X \in M_m \otimes M_n$ is called **separable** if it can be written as

$$X = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|, \quad p_i \geq 0 \; \forall \, i.$$

  If we let S_m be the cone of separable operators in $M_m \otimes M_n$, then $\{S_m\}$ is an operator system (which we will denote by $OMAX$).

- An operator $X \in M_m \otimes M_n$ is called **block positive** if

$$(\langle a| \otimes \langle b|)X(|a\rangle \otimes |b\rangle) \geq 0 \quad \text{for all} \quad |a\rangle \in \mathbb{C}^m, \; |b\rangle \in \mathbb{C}^n.$$

  If we let P_m be the cone of block positive operators in $M_m \otimes M_n$, then $\{P_m\}$ is an operator system (which we will denote by $OMIN$).

Positive Maps

A map on M_n is called **positive** if it preserves positive semidefiniteness. That is, $\Phi \in \mathcal{L}(M_n)$ is positive if $\Phi(X) \in M_n^+$ whenever $X \in M_n^+$.

- Positive maps play an important role in quantum information theory (particularly in entanglement theory).
- We will denote the set of postive maps by $\mathcal{P}$.
- We often want a stronger condition than positivity when dealing with operator systems though. We don't just want the cones C_1 to be preserved, but we want *all* of the cones $\{C_m\}$ to be preserved.

Completely Positive Maps

Suppose we are given operator systems O_1 and O_2, defined by cones $\{C_m\}$ and $\{D_m\}$, respectively.

- A map $\Phi \in \mathcal{L}(M_n)$ is called **completely positive from O_1 to O_2** if

$$(id_m \otimes \Phi)(C_m) \subseteq D_m \quad \text{for all } m.$$

- The set of completely positive maps from O_1 to O_2 will be denoted by $\mathcal{CP}(O_1, O_2)$.
- $\mathcal{CP}(M_n, M_n)$ is the usual set of completely positive maps that quantum information theorists know and love. For brevity, we will denote it simply by $\mathcal{CP}$.

Uniqueness of Operator Systems

The main workhorse that allows us to deal with operator systems is the following result, which says that they are uniquely determined by the *n*th cone – that is, the cone $C_n \subset M_n \otimes M_n$.

Proposition

Let $C_n \subseteq M_n \otimes M_n$ be a convex cone such that

- $S_n \subseteq C_n \subseteq P_n$*; and*
- $(\mathrm{Ad}_A \otimes id_n)(C_n) \subseteq C_n$ *for all* $A \in M_n$*.*

Then there exists a unique family of cones $\{C_m\}_{m \neq n}$ *such that* $\{C_m\}_{m=1}^{\infty}$ *defines an operator system on* M_n*.*

Uniqueness of Operator Systems

Corollary

Let $\Phi : M_n \to M_n$ and let O_1 and O_2 be operator systems defined by families of cones $\{C_m\}_{m=1}^{\infty}$ and $\{D_m\}_{m=1}^{\infty}$, respectively. Then $\Phi \in \mathcal{CP}(O_1, O_2)$ if and only if $(id_n \otimes \Phi)(C_n) \subseteq D_n$.

The above result generalizes the well-known result of Choi that says that Φ is completely positive if and only if it is *n*-positive.

What is a Right-CP-Invariant Cone?

Observe that if $\Phi_1 \in \mathcal{CP}(O_1, O_2)$ and $\Phi_2 \in \mathcal{CP}(O_2, O_3)$, then $\Phi_2 \circ \Phi_1 \in \mathcal{CP}(O_1, O_3)$.

- In particular, if $\Phi_1 \in \mathcal{CP}$ and $\Phi_2 \in \mathcal{CP}(M_n, O)$, then $\Phi_2 \circ \Phi_1 \in \mathcal{CP}(M_n, O)$.
- We use the term **right-CP-invariant** for cones $\mathcal{C} \subseteq \mathcal{L}(M_n)$ with the property that $\mathcal{C} \circ \mathcal{CP} = \mathcal{C}$.

Left-CP-Invariant and Mapping Cones

We just saw that $\mathcal{CP}(M_n, O)$ is always right-CP-invariant.

- Similarly, $\mathcal{CP}(O, M_n)$ is always **left-CP-invariant**.
- A cone that is both left-CP-invariant and right-CP-invariant is called a **mapping cone** – such cones are well-studied for independent reasons in operator theory.
- Examples of mapping cones include the cones of positive, completely positive, and entanglement-breaking maps (denoted by $\mathcal{P}$, $\mathcal{CP}$, and $\mathcal{S}$, respectively).

Operator Systems ↔ Right-CP-Invariant Cones

Recall that $\mathcal{CP}(M_n, O)$ is always right-CP-invariant. It turns out that right-CP-invariance completely characterizes the possible cones of completely positive maps.

Theorem

Let $\mathcal{C} \subseteq \mathcal{L}(M_n)$ be a convex cone. The following are equivalent:

- *$\mathcal{C}$ is right-CP-invariant with $\mathcal{S} \subseteq \mathcal{C} \subseteq \mathcal{P}$.*
- *There exists an operator system O such that $\mathcal{C} = \mathcal{CP}(M_n, O)$.*

Furthermore, if O is defined by the cones $\{C_m\}_{m=1}^{\infty}$ then C_n is the cone of Choi matrices of maps in $\mathcal{C}$. Thus, O is uniquely determined by $\mathcal{C}$.

Basic Examples

The previous theorem established a bijection between operator systems and right-CP-invariant cones, and furthermore showed that bijection is essentially the same as the Choi-Jamiołkowski isomorphism.

- The "naïve" operator system M_n corresponds to the cone of "standard" completely positive maps $\mathcal{CP}(M_n, M_n)$.
- The operator system $OMIN$, defined by the cones of block-positive operators $\{P_m\}$, corresponds to the cone of positive maps $\mathcal{P}$. $OMIN$ is the largest possible operator system.
- The operator system $OMAX$, defined by the cones of separable operators $\{S_m\}$, corresponds to the cone of entanglement-breaking maps $\mathcal{S}$. $OMAX$ is the smallest possible operator system.

k-Positive Maps

A linear map $\Phi \in \mathcal{L}(M_n)$ is called ***k*-positive** if $id_k \otimes \Phi$ is positive.

- k-positive maps play a role in entanglement theory analogous to the role of positive maps – they help distinguish states with Schmidt number $\leq k$ from those with Schmidt number $> k$.
- The cone of k-positive maps is right-CP-invariant. The cones that define the corresponding operator system are the cones of ***k*-block positive** operators. That is, operators such that $\langle v|X|v\rangle \geq 0$ whenever the Schmidt rank of $|v\rangle$ is $\leq k$.
- This operator system is called the **super *k*-minimal** operator system, denoted $OMIN_k$. It is the largest operator system $\{C_m\}$ such that $C_m = M_{mn}^+$ for $1 \leq m \leq k$.

k-Entanglement Breaking Maps

A linear map $\Phi \in \mathcal{L}(M_n)$ is called **k-entanglement breaking** if $(id_n \otimes \Phi)(X)$ has Schmidt number $\leq k$ for any $X \in M_{n^2}^+$.

- The cone of k-entanglement breaking maps is right-CP-invariant. The operator system associated with the cone of k-entanglement breaking maps is defined by the cones of operators with Schmidt rank $\leq k$.
- This operator system is called the **super k-maximal** operator system, denoted $OMAX_k$. It is the smallest operator system $\{C_m\}$ such that $C_m = M_{mn}^+$ for $1 \leq m \leq k$.

Anti-Degradable Maps

A map $\Phi \in \mathcal{CP}$ is called **anti-degradable** if there exists $\Psi \in \mathcal{CP}$ such that $\Psi \circ \Phi^C = \Phi$, where Φ^C is the complementary map of Φ.

- The cone of anti-degradable maps is convex and right-CP-invariant.
- This cone is *not* left-CP-invariant.
- The operator system it gives rise to does not seem to be one that has been studied in operator theory.

Anti-Degradable Maps

An operator $X \in (M_m \otimes M_n)^+$ is called **shareable** if there exists $\tilde{X} \in (M_m \otimes M_n \otimes M_n)^+$ such that $\mathrm{Tr}_2(\tilde{X}) = \mathrm{Tr}_3(\tilde{X}) = X$. We will denote by $H_m \subset M_m \otimes M_n$ the set of shareable operators.

- The Choi matrix of an anti-degradable map is a shareable operator, and a shareable operator is always the Choi matrix of some anti-degradable map.
- The operator system associated with the cone of anti-degradable maps is $\{H_m\}$.

Local k-Broadcasting Maps

A map $\Phi \in \mathcal{CP}$ is called **local k-broadcasting** if there exists a completely positive $\tilde{\Phi} : M_n \to M_n^{\otimes k}$ with the property that $\Phi = \mathrm{Tr}_{\bar{i}} \circ \tilde{\Phi}$ for all $1 \leq i \leq k$.

- In the $k = 2$ case, these are exactly the anti-degradable maps (slightly non-trivial, but not horribly so).
- These cones, like the cone of anti-degradable maps, are convex and right-CP-invariant.
- These cones are *not* left-CP-invariant (except in the trivial $k = 1$ and $k = \infty$ cases, in which case we get the cones of completely positive and entanglement breaking maps, respectively).

Local k-Broadcasting Maps

An operator $X \in (M_m \otimes M_n)^+$ is called **k-shareable** if there exists $\tilde{X} \in (M_m \otimes M_n^{\otimes k})^+$ such that $\mathrm{Tr}_{\bar{i}}(\tilde{X}) = X$ for all $2 \leq i \leq k+1$. We will denote by $H_m^k \subset M_m \otimes M_n$ the set of k-shareable operators.

- The Choi matrix of a local k-broadcasting map is a k-shareable operator, and a k-shareable operator is always the Choi matrix of some local k-broadcasting map.
- The operator system associated with the cone of local k-broadcasting maps is $\{H_m^k\}$.
- These operator systems provide a natural (infinite) hierarchy that interpolates between M_n and $OMAX$.

Open Questions

1. What other "natural" convex cones are right-CP-invariant?

   a) A possible example is the cone of **locally entanglement-annihilating** maps: maps Φ such that $(\Phi \otimes \Phi)(X) \in S_n$ for all $X \in (M_n \otimes M_n)^+$. This cone is right-CP-invariant (and left-CP-invariant), but is it convex?

   b) What about the set of entanglement binding maps? This set is right-CP-invariant. However, it is convex (and hence corresonds to an operator system) if and only if all bound entangled states are PPT (decade-old open problem).

2. This seems to be a strong link between quantum information theory and operator theory. Find applications going in either direction!

Further Reading

- N. J., D. W. Kribs, V. I. Paulsen, and R. Pereira, *Minimal and maximal operator spaces and operator systems in entanglement theory*. J. Funct. Anal. **260** 8, 2407-2423 (2011). E-print: arXiv:1010.1432 [math.OA]

- N. J. and E. Størmer, *Mapping cones are operator systems*. Preprint (2011). E-print: arXiv:1102.2012 [math.OA]