Right CP-Invariant Cones of Superoperators

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Choi Matrices

We will be mostly concerned with the space of $n \times n$ complex matrices M_n and the space of linear maps on M_n: $\mathcal{L}(M_n)$.

Given a linear map $\Phi \in \mathcal{L}(M_n)$, its **Choi matrix** is defined by

$$C_\Phi := \sum_{ij=1}^{n} E_{ij} \otimes \Phi(E_{ij}),$$

where $\{E_{ij}\}$ is the family of standard matrix units in M_n.

Given a set $\mathcal{K} \subseteq \mathcal{L}(M_n)$, we define

$$C_\mathcal{K} := \big\{ C_\Phi : \Phi \in \mathcal{K} \big\}.$$

What are Cones?

A **cone** $K \subset M_n$ is a set of Hermitian operators such that $\lambda K = K$ for all $\lambda \geq 0$.

Similarly, a set $\mathcal{K} \subset \mathcal{L}(M_n)$ is a **cone** if $C_{\mathcal{K}}$ is a cone.

Equivalently, $\mathcal{K} \subset \mathcal{L}(M_n)$ is a cone if it is a set of Hermiticity-preserving maps (i.e., maps Φ such that $\Phi(X)^\dagger = \Phi(X)$ whenever $X^\dagger = X$) with the property that $\lambda\mathcal{K} = \mathcal{K}$ for all $\lambda \geq 0$.

Positive Maps

A map Φ on M_n is called **positive** if $\Phi(X) \in M_n^+$ whenever $X \in M_n^+$.

- Positive maps play an important role in quantum information theory (particularly in entanglement theory).
- We denote the set of positive maps by $\mathcal{P}$.
- $\mathcal{P}$ is a convex cone.

Completely Positive Maps

If $id_m \otimes \Phi$ is positive for all $m \geq 1$, then Φ is called **completely positive** (CP).

- A well-known result of Choi (1975) says that Φ is CP $\Leftrightarrow$ $C_\Phi \in (M_n \otimes M_n)^+$ $\Leftrightarrow$ there exist operators $\{A_i\}$ such that

$$\Phi(X) = \sum_i A_i X A_i^*.$$

- We denote the set of completely positive maps by $\mathcal{CP}$.
- $\mathcal{CP}$ is a convex cone.

Superpositive Maps

If Φ can be written in the form

$$\Phi(X) = \sum_i A_i X A_i^*$$

with $\mathrm{rank}(A_i) = 1$ for all i, then Φ is called **superpositive**.

- We denote the set of superpositive maps by $\mathcal{S}$.
- $\mathcal{S}$ is a convex cone.

What is an Operator System?

An (abstract) **operator system on M_n** is a family of convex cones $K_m \subseteq M_m \otimes M_n$ (one cone for each $m \geq 1$) that satisfy two properties:

1. $K_1 = M_n^+$, the cone of positive-semidefinite operators in M_n; and
2. for each $m_1, m_2 \geq 1$ and $A \in M_{m_1,m_2}$ we have $(A \otimes I)K_{m_1}(A^* \otimes I) \subseteq K_{m_2}$.

Intuitively, these restrictions say that each cone K_m is somehow "like" the cone of positive semidefinite operators.

Examples of Operator Systems on M_n

The most natural operator system on M_n is the one that arises by making the association $M_m \otimes M_n \cong M_{mn}$ in the natural way and letting $K_m = M_{mn}^+$; the cone of positive semidefinite operators. Keep this “naïve” operator system in mind!

We denote the “naïve” operator system simply by M_n. We will denote other general operator systems on M_n by things like $O_1(M_n)$ and $O_2(M_n)$ (or simply O_1 and O_2).

Examples of Operator Systems on M_n

An operator $X \in M_m \otimes M_n$ is called **separable** if it can be written in the form

$$X = \sum_i A_i \otimes B_i \quad \text{with} \quad A_i \in M_m^+, B_i \in M_n^+ \ \forall\, i.$$

If we let S_m be the cone of separable operators in $M_m \otimes M_n$, then $\{S_m\}$ is an operator system.

Also, $S_n = C_{\mathcal{S}}$.

Examples of Operator Systems on M_n

An operator $X \in M_m \otimes M_n$ is called **block positive** if

$$(\mathbf{a} \otimes \mathbf{b})^* X (\mathbf{a} \otimes \mathbf{b}) \geq 0 \quad \text{for all} \quad \mathbf{a} \in \mathbb{C}^m, \ \mathbf{b} \in \mathbb{C}^n.$$

If we let P_m be the cone of block positive operators in $M_m \otimes M_n$, then $\{P_m\}$ is an operator system.

Also, $P_n = C_{\mathcal{P}}$.

What is a Right CP-Invariant Cone?

We use the term **right CP-invariant** for cones $\mathcal{K}$ with the property that $\mathcal{K} \circ \mathcal{CP} \subseteq \mathcal{K}$.

- We can analogously define a **left CP-invariant** cone $\mathcal{K}$ to be one with the property that $\mathcal{CP} \circ \mathcal{K} \subseteq \mathcal{K}$.
- The cones $\mathcal{P}$, $\mathcal{CP}$, and $\mathcal{S}$ are right (and left) CP-invariant.

Dual Cones

We define the **dual** of a cone $\mathcal{K} \subseteq \mathcal{L}(M_n)$ via Choi matrices:

$$\mathcal{K}^\circ := \big\{ \Phi \in \mathcal{L}(M_n) : \mathrm{Tr}(C_\Phi C_\Psi) \geq 0 \ \ \forall \, \Psi \in \mathcal{K} \big\}.$$

- $\mathcal{CP}^\circ = \mathcal{CP}$.
- $\mathcal{P}^\circ = \mathcal{S}$.

Properties of Right CP-Invariant Cones

Proposition

Let $\mathcal{K} \subseteq \mathcal{L}(M_n)$ be a right CP-invariant cone. The following are equivalent:

(a) $\Phi \in \mathcal{K}$; *and*

(b) $(id_n \otimes \Phi)(X) \in C_{\mathcal{K}}$ *for all* $X \in (M_n \otimes M_n)^+$.

- Special case: if $\mathcal{K} = \mathcal{S}$ then this says that $\Phi \in \mathcal{S}$ if and only if $(id_n \otimes \Phi)(X)$ is separable for all $X \in (M_n \otimes M_n)^+$.
- For this reason, superpositive maps are sometimes called **entanglement-breaking maps**.

Properties of Right CP-Invariant Cones

Proposition

If $\mathcal{K} \subseteq \mathcal{L}(M_n)$ is a right CP-invariant cone then so is $\mathcal{K}^\circ$.

Proposition

Let $\mathcal{K} \subseteq \mathcal{L}(M_n)$ be a right CP-invariant cone and let $\Phi \in \mathcal{L}(M_n)$. The following are equivalent:

(a) $\Phi \in \mathcal{K}^\circ$; *and*

(b) $\Omega^\dagger \circ \Phi$ *is completely positive for all* $\Omega \in \mathcal{K}$.

Completely Positive Maps

Suppose we are given operator systems O_1 and O_2, defined by cones $\{J_m\}$ and $\{K_m\}$, respectively.

- A map Φ is called **completely positive from O_1 to O_2** if

$$(id_m \otimes \Phi)(J_m) \subseteq K_m \quad \text{for all } m.$$

- We denote the set of completely positive maps from O_1 to O_2 by $\mathcal{CP}(O_1, O_2)$.
- $\mathcal{CP}(M_n, M_n)$ is the usual set of “standard” completely positive maps.

Operator Systems $\leftrightarrow$ Right CP-Invariant Cones

It is easy to see that $\mathcal{CP}(M_n, O)$ is right CP-invariant for any operator system O. In fact, right CP-invariance completely characterizes the possible cones of completely positive maps.

Theorem

Let $\mathcal{K} \subseteq \mathcal{L}(M_n)$ be a convex cone. The following are equivalent:

- *$\mathcal{K}$ is right CP-invariant with $\mathcal{S} \subseteq \mathcal{K} \subseteq \mathcal{P}$.*
- *There exists an operator system O such that $\mathcal{K} = \mathcal{CP}(M_n, O)$.*

Basic Examples

We noted earlier that $\mathcal{P}$, $\mathcal{CP}$, and $\mathcal{S}$ are right CP-invariant. So what are the corresponding operator systems?

- The "naïve" operator system M_n gives the cone of "standard" completely positive maps $\mathcal{CP}$ (by definition).
- The operator system $\{P_m\}$ of block positive operators gives the cone of positive maps $\mathcal{P}$.
- Paulsen, Todorov, and Tomforde (2010) showed that there is a largest operator system $OMIN(M_n)$. It is defined by the cones $\{P_m\}$.

Basic Examples

Similarly, the operator system $\{S_m\}$ of separable operators gives the cone of superpositive maps $\mathcal{S}$.

- Paulsen, Todorov, and Tomforde (2010) also showed that there is a smallest operator system $OMAX(M_n)$. It is defined by the cones $\{S_m\}$.

k-Positive Maps

A linear map $\Phi \in \mathcal{L}(M_n)$ is called **k-positive** if $id_k \otimes \Phi$ is positive. The set of k-positive maps is $\mathcal{P}_k$.

A map is called **k-superpositive** if it can be written in the form

$$\Phi(X) = \sum_i A_i X A_i^*$$

with $\mathrm{rank}(A_i) \leq k$ for all i. The set of k-superpositive maps is $\mathcal{S}_k$.

$\mathcal{P}_k$ and $\mathcal{S}_k$ are right (and left) CP-invariant.

Complementary Maps

Recall the Stinespring dilation theorem, which says that for any CP map $\Phi \in \mathcal{L}(M_n)$ there exists $A : \mathbb{C}^n \rightarrow \mathbb{C}^{n^2} \otimes \mathbb{C}^n$ so that:

$$\Phi(X) = \mathrm{Tr}_1(AXA^*).$$

The **complementary map** of Φ is defined by

$$\Phi^C(X) := \mathrm{Tr}_2(AXA^*).$$

Anti-Degradable Maps

A map $\Phi \in \mathcal{CP}$ is called **anti-degradable** if there exists $\Psi \in \mathcal{CP}$ such that $\Psi \circ \Phi^C = \Phi$.

- Intuitively, these maps leak more information than they preserve.
- The cone of anti-degradable maps is convex and right CP-invariant.
- This cone is *not* left CP-invariant.

Shareable Operators

An operator $X \in (M_m \otimes M_n)^+$ is called **shareable** if there exists $\tilde{X} \in (M_m \otimes M_n \otimes M_n)^+$ such that $\mathrm{Tr}_2(\tilde{X}) = \mathrm{Tr}_3(\tilde{X}) = X$.

- We use H_m to denote the cone of shareable operators in $M_m \otimes M_n$.
- The set of cones $\{H_m\}$ forms an operator system, which we denote O_H.
- $\mathcal{CP}(M_n, O_H)$ is the set of anti-degradable maps.

Thank you!

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