Norms and Cones in the Theory of Quantum Entanglement

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We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $F$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- $M_n$, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B \rangle := \text{Tr}(A^\dagger B); \text{ and}$$

- $M_n^H$, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.
If $\mathcal{H}$ is a real Hilbert space, then a **cone** $C \subseteq \mathcal{H}$ is a set satisfying

$$v \in C \implies \lambda v \in C \quad \forall \lambda \geq 0.$$ 

$C$ is **convex** if $\lambda v + (1 - \lambda)w \in C$ whenever $v, w \in C$ and $0 \leq \lambda \leq 1$.

For example, the set of positive semidefinite matrices $M_n^+ \subset M_n^H$ is a cone:

$$A \in M_n^+ \iff \langle v | A | v \rangle \geq 0 \quad \forall |v\rangle \in \mathbb{C}^n.$$
The **dual** of a cone $C$ on $\mathcal{H}$ is defined as follows:

$$C^\circ := \{ v \in \mathcal{H} : \langle w | v \rangle \geq 0 \ \forall \ w \in C \}.$$  

- $C^\circ$ is always closed and convex, even if $C$ isn’t.
- If $C$ is closed and convex then $C^{\circ\circ} = C$.
- The positive semidefinite cone is self-dual (i.e., $(M_n^+)^\circ = M_n^+$).
A norm on $\mathcal{H}$ is a function $\| \cdot \| : \mathcal{H} \to \mathbb{R}$ satisfying the following three properties:

- $\|v\| \geq 0$ for all $v \in \mathcal{H}$, with equality if and only if $v = 0$;
- $\|cv\| = |c|\|v\|$ for all $c \in \mathbb{F}, v \in \mathcal{H}$;
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in \mathcal{H}$.

The inner product induces a norm on any Hilbert space: $\sqrt{\langle v | v \rangle}$. For $\mathbb{C}^n$, this is the Euclidean norm $\| \cdot \|$. For $M_n$, this is the Frobenius norm $\| \cdot \|_F$. 
Two other important norms on $M_n$ include:

- the **operator norm**

$$
\|A\| := \sup \left\{ |\langle v|A|w\rangle| \right\} = \sigma_1(A), \text{ and}
$$

- the **trace norm**

$$
\|A\|_{tr} := \sum_{i=1}^{n} \sigma_i(A).
$$
The **dual** of a norm $\| \cdot \|$ on $\mathcal{H}$ is defined as follows:

$$
\|v\|^\circ := \sup_{w \in \mathcal{H}} \{ |\langle w | v \rangle| : \|w\| \leq 1 \}.
$$

- The norm induced by the inner product is its own dual, and
- the operator norm and the trace norm are duals of each other.
A **pure quantum state** is a unit vector $|v\rangle \in \mathbb{C}^n$.

A **mixed quantum state** is a positive semidefinite matrix $\rho \in M_n^H$ with $\text{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle \langle v_i|.$$
We often work with the tensor product of quantum systems. A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called separable if we can find $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$ so that

$$|v\rangle = |a\rangle \otimes |b\rangle.$$ 

A mixed state $\rho \in M_m^H \otimes M_n^H$ is called separable if it can be written as a convex combination of separable pure states:

$$\rho = \sum_i p_i |v_i\rangle \langle v_i| \text{ with each } |v_i\rangle \text{ separable.}$$
Schmidt Decomposition Theorem

**Theorem (Schmidt decomposition)**

For each $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ there exists:

- a positive integer $k \leq \min\{m, n\}$;
- positive real constants $\{\alpha_i\}_{i=1}^k$ with $\sum_{i=1}^k \alpha_i^2 = 1$; and
- orthonormal sets $\{|a_i\rangle\}_{i=1}^k \subset \mathbb{C}^m$ and $\{|b_i\rangle\}_{i=1}^k \subset \mathbb{C}^n$

such that

$$|v\rangle = \sum_{i=1}^k \alpha_i |a_i\rangle \otimes |b_i\rangle.$$
The integer $k$ is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^{k}$ are called the **Schmidt coefficients** of $|v\rangle$.
- Schmidt rank and Schmidt coefficients are easy to calculate.
The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least $k$ such that it can be written as a convex combination of pure states with Schmidt rank no larger than $k$:

$$\rho = \sum_i p_i |v_i\rangle \langle v_i| \text{ with } SR(|v_i\rangle) \leq k \text{ for all } i.$$ 

- $SN(\rho) = 1$ if and only if $\rho$ is separable.
- If $SN(\rho) \geq 2$ then $\rho$ is called **entangled**.
- $SN(|v\rangle \langle v|) = SR(|v\rangle)$ for all $|v\rangle$.
- Schmidt number is difficult to calculate in general.
Block Positivity

The set $S_k$ of (positive scalar multiples of) mixed states with Schmidt number $\leq k$ is a closed, convex cone.

- For any $\rho \notin S_k$, there exists $X \in M_m^H \otimes M_n^H$ such that $\text{Tr}(X\sigma) \geq 0$ for all $\sigma \in S_k$ and $\text{Tr}(X\rho) < 0$.

- Such a matrix $X$ is called a $k$-entanglement witness.

- If we require just the first property (i.e., $\text{Tr}(X\sigma) \geq 0$ for all $\sigma \in S_k$), then we call $X$ $k$-block positive.
The set of $k$-block positive operators is a closed, convex cone. It is (by definition) the dual of $S_k$. 

\[ \text{Positive semidefinite} \]
\[ \text{Schmidt number } \leq n-1 \]
\[ \text{Schmidt number } \leq k \]
\[ \text{Schmidt number } \leq 2 \]
\[ \text{Separable} \]
\[ \text{Block positive} \]
\[ \text{2-block positive} \]
\[ \text{k-block positive} \]
\[ \text{(n-1)-block positive} \]
\[ \text{Positive semidefinite} \]
We now introduce a family of norms that characterize \( k \)-block positivity. For \( X \in M_m \otimes M_n \) we define

\[
\|X\|_{S(k)} := \sup_{|v\rangle, |w\rangle} \left\{ |\langle w | X | v\rangle| : \text{SR}(|v\rangle), \text{SR}(|w\rangle) \leq k \right\}.
\]

- \( \|X\|_{S(1)} \leq \|X\|_{S(2)} \leq \cdots \leq \|X\|_{S(\min\{m,n\})} = \|X\| \)
- Any \( Y \in M_m^H \otimes M_n^H \) can be written in the form \( Y = c I - X \) for some \( X \in (M_m \otimes M_n)^+ \). Then \( Y \) is \( k \)-block positive if and only if \( c \leq \|X\|_{S(k)} \).
The dual of the $S(k)$-norm has the following form:

$$\|X\|_{S(k)}^\circ = \inf \left\{ \sum_i |c_i| : X = \sum_i c_i |v_i\rangle\langle w_i| \right\}$$

with $SR(|v_i\rangle), SR(|w_i\rangle) \leq k \ \forall \ i$.

- $\|X\|_{S(1)}^\circ \geq \|X\|_{S(2)}^\circ \geq \cdots \geq \|X\|_{S(\min\{m,n\})}^\circ = \|X\|_{tr}$
- If $\rho$ is a density operator, then $\|\rho\|_{S(k)}^\circ = 1$ if and only if $SN(\rho) \leq k$. 
Values on Pure States

Schmidt number is easy to determine for pure states, so we might hope that $\| \cdot \|_{S(k)}$ and $\| \cdot \|_{S(k)}^\circ$ are easy to compute for pure states too.

Suppose $|v\rangle$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Then

$$\| |v\rangle \langle v| \|_{S(k)} = \sum_{i=1}^{k} \alpha_i^2.$$
Similarly, let \( r \) be the largest index \( 1 \leq r < k \) such that
\[
\alpha_r > \sum_{i=r+1}^{\min\{m,n\}} \frac{\alpha_i}{(k - r)}
\]
(or take \( r = 0 \) if no such index exists) and define
\[
\tilde{\alpha} := \sum_{i=r+1}^{\min\{m,n\}} \frac{\alpha_i}{(k - r)}.
\]
Then
\[
\|\langle v | v \rangle\|_{S(k)}^\circ = \sum_{i=1}^{r} \alpha_i^2 + (k - r)\tilde{\alpha}^2.
\]

When \( k = 1 \), this simplifies to
\[
\|\langle v | v \rangle\|_{S(1)}^\circ = \left( \sum_{i=1}^{\min\{m,n\}} \alpha_i \right)^2.
\]
In general, computing $\| \cdot \|_{S(k)}$ or $\| \cdot \|_{S(k)}^\circ$ is difficult, so we find bounds for them instead:

- $\| X \|_{S(h)} \leq \| X \|_{S(k)} \leq \frac{k}{h} \| X \|_{S(h)}$ when $h \leq k$.
- $\| X \|_{S(k)} \geq \frac{k \lambda_{mn-(n-h)(m-h)}}{h}$ when $h \geq k$.
- $\| X \|_{S(k)} \leq \sum_i \sum_{j=1}^{k} \lambda_i \alpha_{ij}^2$.
- $\| X \|_{S(k)} \leq \| R(X) \|_{(k^2,2)}$.
- $\| X \|_{S(k)} \geq \| (id_m \otimes \mathcal{E}^\dagger)(X) \|_{S(k)}$ for all quantum channels $\mathcal{E}$.
Inequalities

\[ \|X\|_{S(1)} \geq \frac{1}{mn} \left( \text{Tr}(X) + \sqrt{\frac{mn \text{Tr}(X^2) - \text{Tr}(X)^2}{mn-1}} \right). \]

\[ \|P\|_{S(k)} \geq \|P\|_{S(h)} + \frac{k-h}{\min\{m,n\}-h} \left( 1 - \|P\|_{S(h)} \right) \text{ when } h \geq k. \]

\[ \|P\|_{S(k)} \geq \min \left\{ 1, \frac{k}{\left\lfloor \frac{1}{2} \left( n+m-\sqrt{(n-m)^2+4r-4} \right) \right\rfloor} \right\}, \text{ where } r := \text{rank}(P). \]

\[ \|P\|_{S(k)} \geq \frac{\min\{m,n\}-k}{mn(\min\{m,n\}-1)} \left( r + \sqrt{\frac{mnr-r^2}{mn-1}} \right) + \frac{k-1}{\min\{m,n\}-1}. \]
We now focus on a less obvious place where the $S(k)$-norms come up: distillability.

Two parties share a quantum state $\rho$ and want to perform quantum teleportation. Their first step is to transform their state into a singlet state – they want to distill $\rho$.

- Separable states $\rho$ are undistillable.
- So are states such that $\rho^\Gamma \in (M_m \otimes M_n)^+$ (where $\Gamma$ is the partial transpose).
- What about the converse?
Define $|\psi\rangle := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle \otimes |i\rangle$. There exist other undistillable states in $M_n^H \otimes M_n^H$ if and only if there exists $\alpha \in (1/n, 1/2]$ such that

$$(I - n\alpha |\psi\rangle \langle \psi|) \otimes r$$

is 2-block positive for all $r \geq 1$.

In the $\alpha = 2/n$ case, this can be restated naturally in terms of the $S(2)$-norm...
Define a family of orthogonal projections recursively as follows:

\[ P_{n,1} = |\psi_+\rangle\langle\psi_+| \in M_n \otimes M_n, \]
\[ P_{n,r} = (I - P_{n,1}) \otimes P_{n,r-1} + P_{n,1} \otimes (I - P_{n,r-1}) \quad \forall r \geq 2. \]

The \( \alpha = 2/n \) case of the conjecture holds if and only if

\[ \| P_{n,r} \|_{S(2)} \leq \frac{1}{2} \quad \forall r \geq 1. \]
The best bounds we have are:

\[ \| P_{n,r} \|_{S(2)} \geq \frac{1}{2} - \left( \frac{1}{2} - \frac{1}{n-2} \right) \left( 1 - \frac{2}{n} \right)^r \] and

\[ \| P_{n,r} \|_{S(2)} \leq 1 - \left( 1 - \frac{2}{n} \right)^r. \]

- The lower bound is tight when \( r = 1 \).
- The lower bound is expected to be tight when \( r = 2 \), but this is unknown even when \( n = 4 \).
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