Norms and Cones in the Theory of Quantum Entanglement

Nathaniel Johnston

University of Guelph

June 26, 2012

Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- M_n, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B\rangle := \mathrm{Tr}(A^\dagger B); \text{ and}$$

- M_n^H, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.

Cones

If $\mathcal{H}$ is a real Hilbert space, then a **cone** $\mathcal{C} \subseteq \mathcal{H}$ is a set satisfying

$$\mathbf{v} \in \mathcal{C} \implies \lambda \mathbf{v} \in \mathcal{C} \ \ \forall \lambda \geq 0.$$

$\mathcal{C}$ is **convex** if $\lambda \mathbf{v} + (1 - \lambda)\mathbf{w} \in \mathcal{C}$ whenever $\mathbf{v}, \mathbf{w} \in \mathcal{C}$ and $0 \leq \lambda \leq 1$.

For example, the set of positive semidefinite matrices $M_n^+ \subset M_n^H$ is a cone:

$$A \in M_n^+ \iff \langle v|A|v\rangle \geq 0 \ \ \forall |v\rangle \in \mathbb{C}^n.$$

Dual Cones

The **dual** of a cone $\mathcal{C}$ on $\mathcal{H}$ is defined as follows:

$$\mathcal{C}^\circ := \big\{ \mathbf{v} \in \mathcal{H} : \langle \mathbf{w} | \mathbf{v} \rangle \geq 0 \ \ \forall\, \mathbf{w} \in \mathcal{C} \big\}.$$

- $\mathcal{C}^\circ$ is always closed and convex, even if $\mathcal{C}$ isn't.
- If $\mathcal{C}$ is closed and convex then $\mathcal{C}^{\circ\circ} = \mathcal{C}$.
- The positive semidefinite cone is self-dual (i.e., $(M_n^+)^\circ = M_n^+$).

Norms

A **norm** on $\mathcal{H}$ is a function $\|\cdot\| : \mathcal{H} \to \mathbb{R}$ satisfying the following three properties:

- $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathcal{H}$, with equality if and only if $\mathbf{v} = 0$;
- $\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$ for all $c \in \mathbb{F}, \mathbf{v} \in \mathcal{H}$;
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}, \mathbf{w} \in \mathcal{H}$.

The inner product induces a norm on any Hilbert space: $\sqrt{\langle \mathbf{v} | \mathbf{v} \rangle}$. For $\mathbb{C}^n$, this is the **Euclidean norm** $\|\cdot\|$. For M_n, this is the **Frobenius norm** $\|\cdot\|_F$.

Examples of Norms

Two other important norms on M_n include:

- the **operator norm**

$$\big\|A\big\| := \sup\Big\{\big|\langle v|A|w\rangle\big|\Big\} = \sigma_1(A), \text{ and}$$

- the **trace norm**

$$\big\|A\big\|_{tr} := \sum_{i=1}^{n} \sigma_i(A).$$

Dual Norms

The **dual** of a norm $|||\cdot|||$ on $\mathcal{H}$ is defined as follows:

$$|||\mathbf{v}|||^\circ := \sup_{\mathbf{w}\in\mathcal{H}} \Big\{ \big|\langle \mathbf{w}|\mathbf{v}\rangle\big| : |||\mathbf{w}||| \leq 1 \Big\}.$$

- The norm induced by the inner product is its own dual, and
- the operator norm and the trace norm are duals of each other.

Quantum States

A **pure quantum state** is a unit vector $|v\rangle \in \mathbb{C}^n$.

A **mixed quantum state** is a positive semidefinite matrix $\rho \in M_n^H$ with $\mathrm{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|.$$

Separability and Entanglement

We often work with the tensor product of quantum systems. A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called separable if we can find $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$ so that

$$|v\rangle = |a\rangle \otimes |b\rangle.$$

A mixed state $\rho \in M_m^H \otimes M_n^H$ is called separable if it can be written as a convex combination of separable pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \text{ with each } |v_i\rangle \text{ separable.}$$

Schmidt Decomposition Theorem

Theorem (Schmidt decomposition)

For each $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ *there exists:*

- *a positive integer* $k \leq \min\{m, n\}$*;*
- *positive real constants* $\{\alpha_i\}_{i=1}^k$ *with* $\sum_{i=1}^k \alpha_i^2 = 1$*; and*
- *orthonormal sets* $\{|a_i\rangle\}_{i=1}^k \subset \mathbb{C}^m$ *and* $\{|b_i\rangle\}_{i=1}^k \subset \mathbb{C}^n$

such that

$$|v\rangle = \sum_{i=1}^k \alpha_i |a_i\rangle \otimes |b_i\rangle.$$

Schmidt Rank

The integer k is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^k$ are called the **Schmidt coefficients** of $|v\rangle$.
- Schmidt rank and Schmidt coefficients are easy to calculate.

Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least k such that it can be written as a convex combination of pure states with Schmidt rank no larger than k:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with } SR(|v_i\rangle) \leq k \text{ for all } i.$$

- $SN(\rho) = 1$ if and only if ρ is separable.
- If $SN(\rho) \geq 2$ then ρ is called **entangled**.
- $SN(|v\rangle\langle v|) = SR(|v\rangle)$ for all $|v\rangle$.
- Schmidt number is difficult to calculate in general.

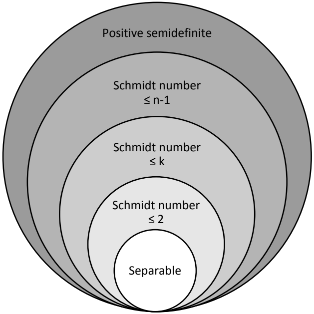
Block Positivity

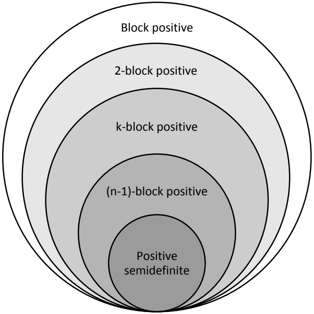
The set $\mathcal{S}_k$ of (positive scalar multiples of) mixed states with Schmidt number $\leq k$ is a closed, convex cone.

- For any $\rho \notin \mathcal{S}_k$, there exists $X \in M_m^H \otimes M_n^H$ such that $\mathrm{Tr}(X\sigma) \geq 0$ for all $\sigma \in \mathcal{S}_k$ and $\mathrm{Tr}(X\rho) < 0$.
- Such a matrix X is called a **k-entanglement witness**.
- If we require just the first property (i.e., $\mathrm{Tr}(X\sigma) \geq 0$ for all $\sigma \in \mathcal{S}_k$), then we call X **k-block positive**.

Duality

The set of k-block positive operators is a closed, convex cone. It is (by definition) the dual of $\mathcal{S}_k$.





$S(k)$-Norms

We now introduce a family of norms that characterize k-block positivity. For $X \in M_m \otimes M_n$ we define

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\}.$$

- $\|X\|_{S(1)} \leq \|X\|_{S(2)} \leq \cdots \leq \|X\|_{S(\min\{m,n\})} = \|X\|$
- Any $Y \in M_m^H \otimes M_n^H$ can be written in the form $Y = cI - X$ for some $X \in (M_m \otimes M_n)^+$. Then Y is k-block positive if and only if $c \leq \|X\|_{S(k)}$.

Duals of the $S(k)$-Norms

The dual of the $S(k)$-norm has the following form:

$$\|X\|_{S(k)}^{\circ} = \inf\Big\{ \sum_i |c_i| : X = \sum_i c_i|v_i\rangle\langle w_i| \text{ with } SR(|v_i\rangle), SR(|w_i\rangle) \leq k \ \forall\, i \Big\}.$$

- $\|X\|_{S(1)}^{\circ} \geq \|X\|_{S(2)}^{\circ} \geq \cdots \geq \|X\|_{S(\min\{m,n\})}^{\circ} = \|X\|_{tr}$
- If ρ is a density operator, then $\|\rho\|_{S(k)}^{\circ} = 1$ if and only if $SN(\rho) \leq k$.

Values on Pure States

Schmidt number is easy to determine for pure states, so we might hope that $\|\cdot\|_{S(k)}$ and $\|\cdot\|_{S(k)}^{\circ}$ are easy to compute for pure states too.

Suppose $|v\rangle$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Then

$$\big\| |v\rangle\langle v| \big\|_{S(k)} = \sum_{i=1}^{k} \alpha_i^2.$$

Values on Pure States

Similarly, let r be the largest index $1 \leq r < k$ such that $\alpha_r > \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$ (or take $r = 0$ if no such index exists) and define $\tilde{\alpha} := \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$. Then

$$\big\|\,|v\rangle\langle v|\,\big\|_{S(k)}^{\circ} = \sum_{i=1}^{r} \alpha_i^2 + (k-r)\tilde{\alpha}^2.$$

When $k = 1$, this simiplifies to

$$\big\|\,|v\rangle\langle v|\,\big\|_{S(1)}^{\circ} = \left(\sum_{i=1}^{\min\{m,n\}} \alpha_i \right)^2.$$

Inequalities

In general, computing $\|\cdot\|_{S(k)}$ or $\|\cdot\|_{S(k)}^{\circ}$ is difficult, so we find bounds for them instead:

- $\|X\|_{S(h)} \leq \|X\|_{S(k)} \leq \frac{k}{h}\|X\|_{S(h)}$ when $h \leq k$.
- $\|X\|_{S(k)} \geq \frac{k\lambda_{mn-(n-h)(m-h)}}{h}$ when $h \geq k$.
- $\|X\|_{S(k)} \leq \sum_i \sum_{j=1}^{k} \lambda_i \alpha_{ij}^2$.
- $\|X\|_{S(k)} \leq \|R(X)\|_{(k^2,2)}$.
- $\|X\|_{S(k)} \geq \|(id_m \otimes \mathcal{E}^\dagger)(X)\|_{S(k)}$ for all quantum channels $\mathcal{E}$.

Inequalities

- $\|X\|_{S(1)} \geq \frac{1}{mn}\left(\mathrm{Tr}(X) + \sqrt{\frac{mn\mathrm{Tr}(X^2) - \mathrm{Tr}(X)^2}{mn-1}}\right)$.
- $\|P\|_{S(k)} \geq \|P\|_{S(h)} + \frac{k-h}{\min\{m,n\}-h}\big(1 - \|P\|_{S(h)}\big)$ when $h \geq k$.
- $\|P\|_{S(k)} \geq \min\left\{1, \frac{k}{\left\lceil \frac{1}{2}\left(n+m-\sqrt{(n-m)^2+4r-4}\right)\right\rceil}\right\}$, where $r := \mathrm{rank}(P)$.
- $\|P\|_{S(k)} \geq \frac{\min\{m,n\}-k}{mn(\min\{m,n\}-1)}\left(r + \sqrt{\frac{mnr-r^2}{mn-1}}\right) + \frac{k-1}{\min\{m,n\}-1}$.

Distillability

We now focus on a less obvious place where the $S(k)$-norms come up: distillability.

Two parties share a quantum state ρ and want to perform quantum teleportation. Their first step is to transform their state into a singlet state – they want to **distill** ρ.

- Separable states ρ are **undistillable**.
- So are states such that $\rho^\Gamma \in (M_m \otimes M_n)^+$ (where Γ is the partial transpose).
- What about the converse?

Distillability

Define $|\psi\rangle := \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|i\rangle \otimes |i\rangle$. There exist other undistillable states in $M_n^H \otimes M_n^H$ if and only if there exists $\alpha \in (1/n, 1/2]$ such that

$$(I - n\alpha|\psi\rangle\langle\psi|)^{\otimes r}$$

is 2-block positive for all $r \geq 1$.

In the $\alpha = 2/n$ case, this can be restated naturally in terms of the $S(2)$-norm...

Distillability

Define a family of orthogonal projections recursively as follows:

$$P_{n,1} = |\psi_+\rangle\langle\psi_+| \in M_n \otimes M_n,$$
$$P_{n,r} = (I - P_{n,1}) \otimes P_{n,r-1} + P_{n,1} \otimes (I - P_{n,r-1}) \quad \forall\, r \geq 2.$$

The $\alpha = 2/n$ case of the conjecture holds if and only if

$$\big\| P_{n,r} \big\|_{S(2)} \leq \frac{1}{2} \;\; \forall\, r \geq 1.$$

Distillability

The best bounds we have are:

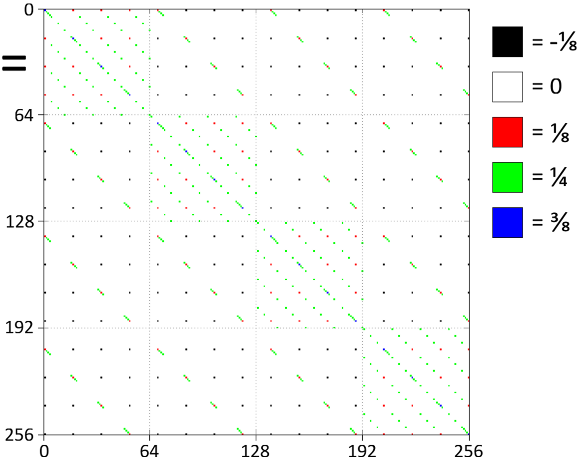
$$\big\|P_{n,r}\big\|_{S(2)} \geq \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{n-2}\right)\left(1 - \frac{2}{n}\right)^r \quad \text{and}$$

$$\big\|P_{n,r}\big\|_{S(2)} \leq 1 - \left(1 - \frac{2}{n}\right)^r.$$

- The lower bound is tight when $r = 1$.
- The lower bound is expected to be tight when $r = 2$, but this is unknown even when $n = 4$.

Distillability

$P_{4,2} =$



Thank You

A big thank-you goes to...

- my advisor: David Kribs;
- my committees: Professors Bauch, Pereira, Poisson, Watrous, and Zeng; and
- all of you for showing up!