On the Minimum Size of Unextendible Product Bases

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June 3, 2013

Unextendible Product Bases

A set $\mathcal{S} \subset \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_p}$ is called an **unextendible product basis (UPB)** if:

- Every $|v\rangle \in \mathcal{S}$ is a **product state** – it can be written in the form

$$|v\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_p\rangle \quad \text{for some} \quad |v_j\rangle \subset \mathbb{C}^{d_j},$$

- For all $|v\rangle, |w\rangle \in \mathcal{S}$, we have $\langle v|w\rangle = 0$, and
- There does not exist a product state $|z\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_p}$ such that $\langle v|z\rangle = 0$ for all $|v\rangle \in \mathcal{S}$.

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Unextendible Product Bases

One of the simplest non-trivial UPBs arises in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, where there is a UPB of 4 states (called "Shifts") [1]:

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \qquad |1\rangle \otimes |+\rangle \otimes |-\rangle$$
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Where $\{|0\rangle, |1\rangle\}$ is the standard basis and $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

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Minimum Size of UPBs

One of the first questions asked about UPBs was what their minimum size is.

A simple lower bound on the size of UPBs in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_p}$ is

$$1 + \sum_{j=1}^{p} (d_j - 1).$$

Proof. We can always construct a state that is orthogonal to any $(d_j - 1)$ states in $\mathbb{C}^{d_j}$.

$\implies$ We can always construct a product state that is orthogonal to any $\sum_{j=1}^{p}(d_j - 1)$ product states in $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_p}$.

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$$1 + \sum_{j=1}^{p} (d_j - 1).$$

We refer to this quantity as the **Trivial Lower Bound (TLB)**.

The Trivial Lower Bound

The question of when the TLB can be attained was solved very quickly [2]:

Theorem (Alon and Lovász)

If $p = 2$ and $\min\{d_1, d_2\} = 2$ then there are no non-trivial UPBs. Otherwise, the TLB is attained if and only if the TLB is even or all d_j's are odd (or both).

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Orthogonality Graphs

Suppose we are given a set of s product states $\mathcal{S} = \{|v_0\rangle, \ldots, |v_{s-1}\rangle\} \subset \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_p}$.

The **orthogonality graph** of $\mathcal{S}$ is the graph on s vertices $(v_0, \ldots, v_{s-1})$ such that there is an edge (v_i, v_j) of color ℓ if and only if $|v_i\rangle$ and $|v_j\rangle$ are orthogonal on party ℓ.

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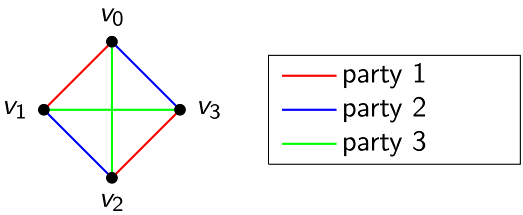

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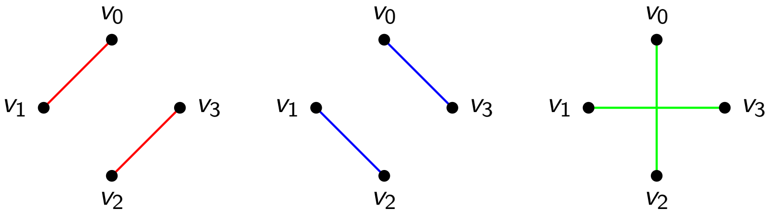

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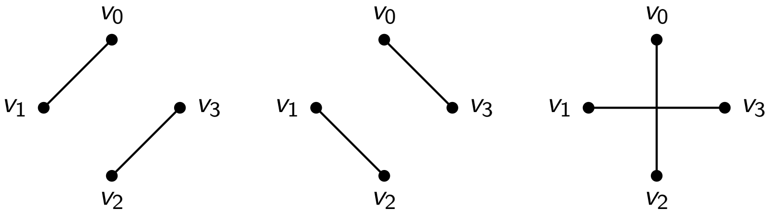

The Bipartite Case: What's Known

We now focus on the bipartite (i.e., $p = 2$) case.

- If $\min\{d_1, d_2\} = 2$ then there are no non-trivial UPBs.
- Alon and Lovász's result tells us that if at least one of d_1 or d_2 is odd, then the smallest UPB has size $d_1 + d_2 - 1$ (this is the TLB).
- Only one other case is known: if $d_1 = d_2 = 4$ then the smallest UPB has size 8 (i.e., TLB + 1) [3].

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The Bipartite Case: What's New

We solve the remaining bipartite cases:

Theorem

If $d_1, d_2 \geq 4$ are both even then the smallest UPB has size $d_1 + d_2$ (i.e., $TLB + 1$).

The Bipartite Case: How It's Done

We already know that the smallest UPB has size at *least* $d_1 + d_2$, so we just need to construct a UPB of this size.

We begin by giving an orthogonality graph, and we then show that there are many sets of product vectors with that orthogonality graph. We then show that some of these sets are unextendible.

We illustrate the construction in the $d_1 = d_2 = 6$ case – it generalizes straightforwardly.

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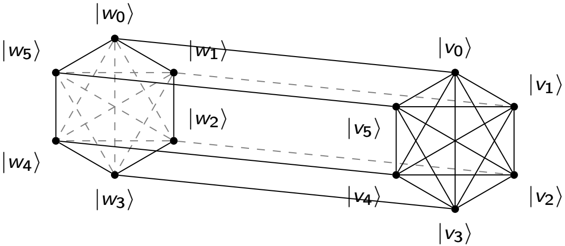

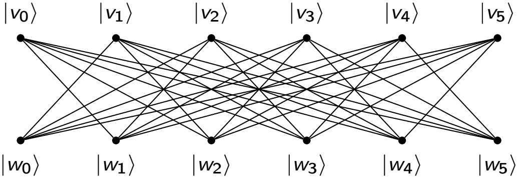

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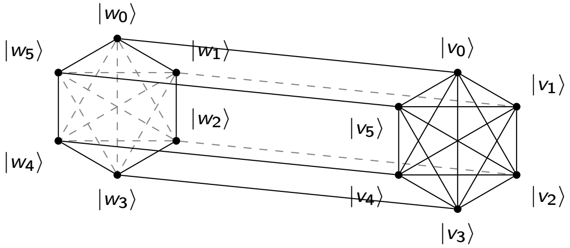

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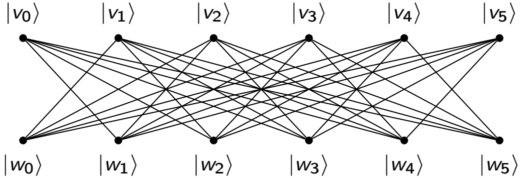

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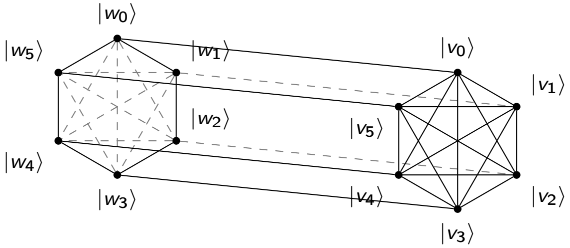

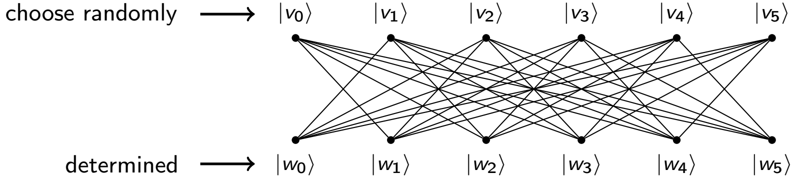

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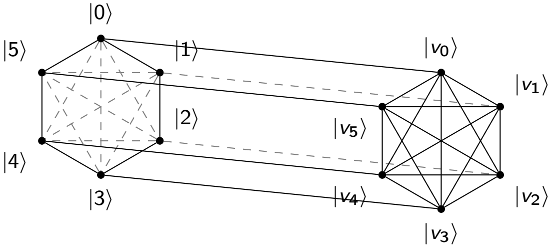

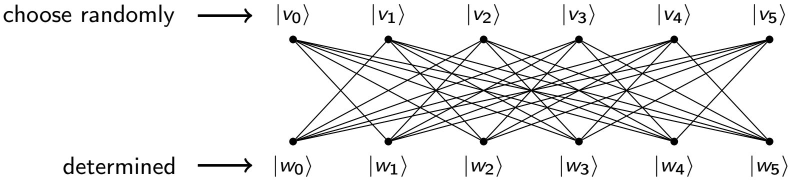

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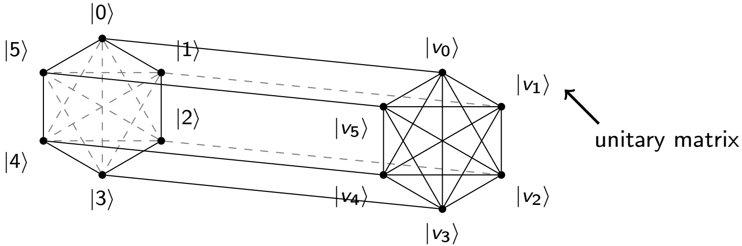

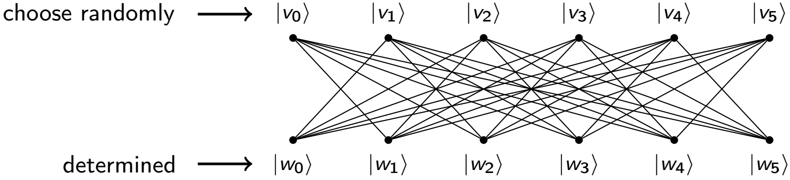

The Qubit Case: What's Known

We now focus on the qubit (i.e., $d_1 = d_2 = \cdots = d_p = 2$) case.

- Alon and Lovász's result tells us that there is a UPB of size $p + 1$ (the TLB) if and only if p is odd.
- It is also known that if $p \equiv 2 \pmod 4$ or $p = 4$ then the smallest UPB is of size $p + 2$ (i.e., TLB + 1). [3]
- What about when $p \geq 8$ is a multiple of 4?

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The Qubit Case: What's New

We solve the remaining qubit cases:

Theorem

If $p = 8$ then the smallest UPB has size 11 (i.e., $TLB + 2$). If $p \geq 12$ is a multiple of 4 then the smallest UPB has size $p + 4$ (i.e., $TLB + 3$).

The Qubit Case: How It's Done

We already know that the smallest UPB has size at least $p + 2$, so we need to rule out UPBs of size $p + 2$ (and $p + 3$ when $p \geq 12$).

- This is messy – we split into about 10 cases and analyze each one individually.
- Actually constructing UPBs of the claimed size is not too difficult though – in the qubit case, we can fairly easily go from an orthogonality graph to a UPB.

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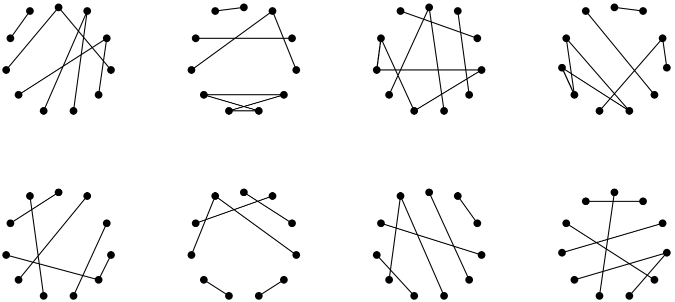

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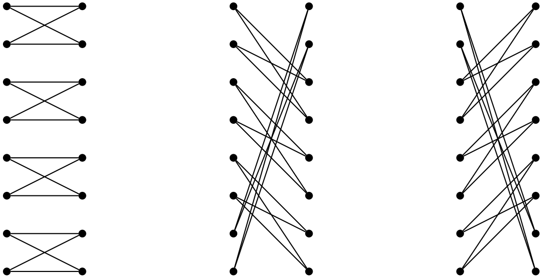

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It's then not difficult to show that this orthogonality graph corresponds to a UPB.

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Outlook and Questions

What is the minimum size of a UPB in general (i.e., when there are many subsystems of arbitrary size)?

What are *all* possible sizes of UPBs? We know minimal and maximal, but which intermediate sizes are achievable? (No, not all intermediate sizes are attainable.)

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