Uniqueness of Quantum States Compatible with Given Measurement Results

J. Chen, H. Dawkins, Z. Ji, **N. Johnston**, D. Kribs, F. Shultz, and B. Zeng

Institute for Quantum Computing, Tuesday Lunch Talk

January 29, 2013

Tomography

We want to determine what state a quantum system is in, so we perform tomography.

i.e., we measure that quantum system repeatedly.

In general, if we are working on a d-dimensional Hilbert space $\mathcal{H}_d$, we need $d^2 - 1$ measurement outcomes to uniquely determine our quantum state.

Tomography

We want to determine what state a quantum system is in, so we perform tomography.

i.e., we measure that quantum system repeatedly.

In general, if we are working on a d-dimensional Hilbert space $\mathcal{H}_d$, we need $d^2 - 1$ measurement outcomes to uniquely determine our quantum state.

Tomography

We want to determine what state a quantum system is in, so we perform tomography.

i.e., we measure that quantum system repeatedly.

In general, if we are working on a d-dimensional Hilbert space $\mathcal{H}_d$, we need $d^2 - 1$ measurement outcomes to uniquely determine our quantum state.

Tomography

We have some tuple of m observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the state ρ, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(\rho) := (\mathrm{Tr}(A_1\rho), \mathrm{Tr}(A_2\rho), \ldots, \mathrm{Tr}(A_m\rho)).$$

We want ρ to be uniquely determined by $\mathbf{A}(\rho)$.

Tomography

We have some tuple of m observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the state ρ, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(\rho) := (\mathrm{Tr}(A_1\rho), \mathrm{Tr}(A_2\rho), \ldots, \mathrm{Tr}(A_m\rho)).$$

We want ρ to be uniquely determined by $\mathbf{A}(\rho)$.

Tomography

We have some tuple of m observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the state ρ, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(\rho) := (\mathrm{Tr}(A_1\rho), \mathrm{Tr}(A_2\rho), \ldots, \mathrm{Tr}(A_m\rho)).$$

We want ρ to be uniquely determined by $\mathbf{A}(\rho)$.

Tomography of Pure States

Question

What if ρ is a pure state?

What do we even mean when we ask for a pure state $\rho = |\phi\rangle\langle\phi|$ to be uniquely determined by $\mathbf{A}(\rho)$?

- **Ideal:** We know that our state is indeed pure.
- **Real world:** Maybe it got mixed.

Tomography of Pure States

Question

What if ρ is a pure state?

What do we even mean when we ask for a pure state $\rho = |\phi\rangle\langle\phi|$ to be uniquely determined by $\mathbf{A}(\rho)$?

Ideal: We know that our state is indeed pure.

Real world: Maybe it got mixed.

Tomography of Pure States

Question

What if ρ is a pure state?

What do we even mean when we ask for a pure state $\rho = |\phi\rangle\langle\phi|$ to be uniquely determined by $\mathbf{A}(\rho)$?

Ideal: We know that our state is indeed pure.

Real world: Maybe it got mixed.

Tomography of Pure States

Question

What if ρ is a pure state?

What do we even mean when we ask for a pure state $\rho = |\phi\rangle\langle\phi|$ to be uniquely determined by $\mathbf{A}(\rho)$?

Ideal: We know that our state is indeed pure.

Real world: Maybe it got mixed.

UDP and UDA

Definition

*A pure state $|\phi\rangle$ is **uniquely determined among pure states (UDP)** for **A** if there does not exist any other pure state with the same measurement results as those of $|\phi\rangle$ when measuring **A**.*

Definition

*A pure state $|\phi\rangle$ is **uniquely determined among all states (UDA)** for **A** if there does not exist any other state (pure or mixed) with the same measurement results as those of $|\phi\rangle$ when measuring **A**.*

UDP and UDA

Definition

*A pure state $|\phi\rangle$ is **uniquely determined among pure states (UDP)** for* **A** *if there does not exist any other pure state with the same measurement results as those of $|\phi\rangle$ when measuring* **A**.

Definition

*A pure state $|\phi\rangle$ is **uniquely determined among all states (UDA)** for* **A** *if there does not exist any other state (pure or mixed) with the same measurement results as those of $|\phi\rangle$ when measuring* **A**.

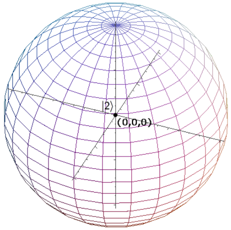
An Example

Consider the $d = 3$ case with the following $m = 3$ observables:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The set of measurement outcomes $\{\mathbf{A}(\rho)\}$ is the unit ball.

$\mathbf{A}(|2\rangle) = (0,0,0)$, and there is no other pure state $|\phi\rangle$ for which $\mathbf{A}(|\phi\rangle) = (0,0,0)$.



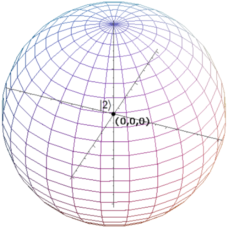
An Example

Consider the $d = 3$ case with the following $m = 3$ observables:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The set of measurement outcomes $\{\mathbf{A}(\rho)\}$ is the unit ball.

$\mathbf{A}(|2\rangle) = (0,0,0)$, and there is no other pure state $|\phi\rangle$ for which $\mathbf{A}(|\phi\rangle) = (0,0,0)$.



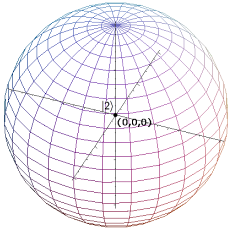
An Example

Consider the $d = 3$ case with the following $m = 3$ observables:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The set of measurement outcomes $\{\mathbf{A}(\rho)\}$ is the unit ball.

$\mathbf{A}(|2\rangle) = (0,0,0)$, and there is no other pure state $|\phi\rangle$ for which $\mathbf{A}(|\phi\rangle) = (0,0,0)$.



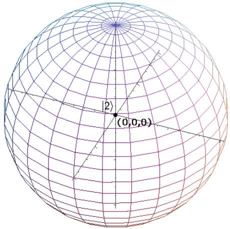
An Example

Consider the $d = 3$ case with the following $m = 3$ observables:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

i.e., the state $|2\rangle$ is UDP for $\mathbf{A}$.

However, $|2\rangle$ is not UDA, since $\mathbf{A}(\rho) = (0,0,0)$ for $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.



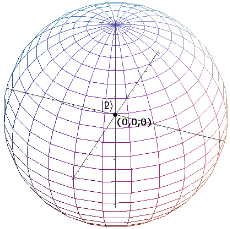
An Example

Consider the $d = 3$ case with the following $m = 3$ observables:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

i.e., the state $|2\rangle$ is UDP for $\mathbf{A}$.

However, $|2\rangle$ is not UDA, since $\mathbf{A}(\rho) = (0,0,0)$ for $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.



Questions

Question

What is the minimum number of observables needed for all pure states to be UDP/UDA?

Question

When does UDP = UDA?

Questions

Question

What is the minimum number of observables needed for all pure states to be UDP/UDA?

Question

When does UDP = UDA?

Some Assumptions

All of the questions we consider depend only on the span of $\{A_1, A_2, \ldots, A_m\}$.

We assume WLOG that the operators $\{A_1, A_2, \ldots, A_m\}$ are traceless and linearly independent.

Some Assumptions

All of the questions we consider depend only on the span of $\{A_1, A_2, \ldots, A_m\}$.

We assume WLOG that the operators $\{A_1, A_2, \ldots, A_m\}$ are traceless and linearly independent.

How many observables suffice for UDP?

In general (i.e., for the case of mixed states), we need $d^2 - 1$ observables to uniquely determine a state.

However, it is known [1] that there exists a family of $4d - 5$ observables so that every pure state is UDP.

The leading term $4d$ is optimal (but the -5 isn't).

[1] *T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. arXiv:1109.5478 [quant-ph], 2011.*

How many observables suffice for UDP?

In general (i.e., for the case of mixed states), we need $d^2 - 1$ observables to uniquely determine a state.

However, it is known [1] that there exists a family of $4d - 5$ observables so that every pure state is UDP.

The leading term $4d$ is optimal (but the -5 isn't).

[1] *T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. arXiv:1109.5478 [quant-ph], 2011.*

How many observables suffice for UDP?

In general (i.e., for the case of mixed states), we need $d^2 - 1$ observables to uniquely determine a state.

However, it is known [1] that there exists a family of $4d - 5$ observables so that every pure state is UDP.

The leading term $4d$ is optimal (but the -5 isn't).

[1] *T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. arXiv:1109.5478 [quant-ph], 2011.*

A Note on the Proof

It was noted in [1] that all pure states are UDP for **A** if and only if the orthogonal complement

$$\mathbf{A}^{\perp} := \{B : B = B^{\dagger}, \mathrm{Tr}(B) = 0, \mathrm{Tr}(A_i B) = 0 \ \forall\, i\}$$

does not contain a matrix of rank ≤ 2.

Thus, if there exists a family of k observables for which every pure state is UDP, then there is a subspace of dimension $d^2 - k - 1$ of Hermitian, traceless matrices of rank ≥ 3 (and vice versa).

[1] *T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. arXiv:1109.5478 [quant-ph], 2011.*

A Note on the Proof

It was noted in [1] that all pure states are UDP for **A** if and only if the orthogonal complement

$$\mathbf{A}^{\perp} := \{B : B = B^{\dagger}, \mathrm{Tr}(B) = 0, \mathrm{Tr}(A_i B) = 0 \ \forall\, i\}$$

does not contain a matrix of rank ≤ 2.

Thus, if there exists a family of k observables for which every pure state is UDP, then there is a subspace of dimension $d^2 - k - 1$ of Hermitian, traceless matrices of rank ≥ 3 (and vice versa).

[1] *T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. arXiv:1109.5478 [quant-ph], 2011.*

A Note on the Proof

Question

What is the maximal dimension of a subspace $\mathcal{S}$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathcal{S}$ has $\mathrm{rank}(X) \geq 3$?

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathcal{S}$ of dimension $d^2 - (4d - 5) - 1 = (d - 2)^2$.

A Note on the Proof

Question

What is the maximal dimension of a subspace $\mathcal{S}$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathcal{S}$ has $\mathrm{rank}(X) \geq 3$?

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathcal{S}$ of dimension $d^2 - (4d - 5) - 1 = (d - 2)^2$.

A Note on the Proof

However, recall that $4d - 5$ isn't optimal – there are cases (e.g., $d = 4$) when we can do better.

In these same situations, we can find a subspace $\mathcal{S}$ of matrices all with rank ≥ 3 of dimension $> (d-2)^2$.

But wait! There is a well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

A Note on the Proof

However, recall that $4d - 5$ isn't optimal – there are cases (e.g., $d = 4$) when we can do better.

In these same situations, we can find a subspace $\mathcal{S}$ of matrices all with rank ≥ 3 of dimension $> (d-2)^2$.

But wait! There is a well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

A Note on the Proof

However, recall that $4d - 5$ isn't optimal – there are cases (e.g., $d = 4$) when we can do better.

In these same situations, we can find a subspace $\mathcal{S}$ of matrices all with rank ≥ 3 of dimension $> (d-2)^2$.

But wait! There is a well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

How Many Observables Suffice for UDA?

We now ask the same question for UDA – what is the fewest number of observables for which every pure state is UDA?

Theorem

There exists a set of $5d - 7$ observables on $\mathcal{H}_d$ for which every pure state is UDA.

How Many Observables Suffice for UDA?

We now ask the same question for UDA – what is the fewest number of observables for which every pure state is UDA?

Theorem

There exists a set of $5d-7$ observables on $\mathcal{H}_d$ for which every pure state is UDA.

Rough Idea of Proof

We have a tuple of observables $(A_1, \ldots, A_m)$ and we want to determine whether or not the pure state $|\phi\rangle$ is UDA.

Any density matrix ρ can be written in the form $\rho = |\phi\rangle\langle\phi| + V$ for some traceless, Hermitian V.

ρ has the same measurement results as $|\phi\rangle$ if and only if $V \in \mathbf{A}^\perp$.

Rough Idea of Proof

We have a tuple of observables $(A_1, \ldots, A_m)$ and we want to determine whether or not the pure state $|\phi\rangle$ is UDA.

Any density matrix ρ can be written in the form $\rho = |\phi\rangle\langle\phi| + V$ for some traceless, Hermitian V.

ρ has the same measurement results as $|\phi\rangle$ if and only if $V \in \mathbf{A}^{\perp}$.

Rough Idea of Proof

We have a tuple of observables $(A_1, \ldots, A_m)$ and we want to determine whether or not the pure state $|\phi\rangle$ is UDA.

Any density matrix ρ can be written in the form $\rho = |\phi\rangle\langle\phi| + V$ for some traceless, Hermitian V.

ρ has the same measurement results as $|\phi\rangle$ if and only if $V \in \mathbf{A}^\perp$.

Rough Idea of Proof

Thus for every pure state to be UDA, we need $|\phi\rangle\langle\phi| + V$ to never be positive semidefinite when $V \in \mathbf{A}^\perp$.

This happens if and only if every element of $\mathbf{A}^\perp$ has at least 2 negative eigenvalues (and hence 2 positive eigenvalues).

How large can such a subspace be?

Rough Idea of Proof

Thus for every pure state to be UDA, we need $|\phi\rangle\langle\phi| + V$ to never be positive semidefinite when $V \in \mathbf{A}^\perp$.

This happens if and only if every element of $\mathbf{A}^\perp$ has at least 2 negative eigenvalues (and hence 2 positive eigenvalues).

How large can such a subspace be?

Rough Idea of Proof

Thus for every pure state to be UDA, we need $|\phi\rangle\langle\phi| + V$ to never be positive semidefinite when $V \in \mathbf{A}^{\perp}$.

This happens if and only if every element of $\mathbf{A}^{\perp}$ has at least 2 negative eigenvalues (and hence 2 positive eigenvalues).

How large can such a subspace be?

Subspaces with Prescribed Eigenvalue Signs

We construct such a subspace of size $(d-3)(d-2)$.

This leads to a family of $(d^2-1)-(d-3)(d-2) = 5d-7$ observables for which every pure state is UDA.

But wait! Once again, this seems to contradict the well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Here we have a subspace of $(d-3)(d-2) > (d-3)^2$ matrices of rank ≥ 4. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

Subspaces with Prescribed Eigenvalue Signs

We construct such a subspace of size $(d-3)(d-2)$.

This leads to a family of $(d^2-1)-(d-3)(d-2) = 5d-7$ observables for which every pure state is UDA.

But wait! Once again, this seems to contradict the well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Here we have a subspace of $(d-3)(d-2) > (d-3)^2$ matrices of rank ≥ 4. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

Subspaces with Prescribed Eigenvalue Signs

We construct such a subspace of size $(d-3)(d-2)$.

This leads to a family of $(d^2-1)-(d-3)(d-2) = 5d-7$ observables for which every pure state is UDA.

But wait! Once again, this seems to contradict the well-known result [2] that says that the largest subspace of matrices of rank $\geq k$ has dimension $(d-k+1)^2$. Here we have a subspace of $(d-3)(d-2) > (d-3)^2$ matrices of rank ≥ 4. Hrm...

[2] *T. S. Cubitt, A. Montanaro, and A. Winter. On the dimension of subspaces with bounded Schmidt rank. J. Math. Phys., 49:022107, 2008.*

Subspaces with Prescribed Ranks

The discrepancy is because we only consider real linear combinations, not complex ones. For example, in the $d = 4$ case, consider the following two matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$

Any real linear combination of A_1 and A_2 has two positive and two negative eigenvalues.

However, $A_1 + iA_2$ has rank 2.

Subspaces with Prescribed Ranks

The discrepancy is because we only consider real linear combinations, not complex ones. For example, in the $d = 4$ case, consider the following two matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$

Any real linear combination of A_1 and A_2 has two positive and two negative eigenvalues.

However, $A_1 + iA_2$ has rank 2.

Subspaces with Prescribed Ranks

The discrepancy is because we only consider real linear combinations, not complex ones. For example, in the $d = 4$ case, consider the following two matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}.$$

Any real linear combination of A_1 and A_2 has two positive and two negative eigenvalues.

However, $A_1 + iA_2$ has rank 2.

Subspaces with Prescribed Ranks

This suggests the following natural questions:

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has rank $\geq k$?

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has at least k positive and k negative eigenvalues?

Subspaces with Prescribed Ranks

This suggests the following natural questions:

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has rank $\geq k$?

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has at least k positive and k negative eigenvalues?

Subspaces with Prescribed Ranks

This suggests the following natural questions:

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has rank $\geq k$?

Question

What is the maximum size of a subspace of Hermitian matrices such that every element has at least k positive and k negative eigenvalues?

When Does UDP = UDA?

There are cases when UDP $\neq$ UDA (e.g., the $d = 3$ case of 7 observables).

When can we guarantee that UDP = UDA (i.e., when can we guarantee that $|\phi\rangle$ being UDP implies it is UDA)?

A complete characterization seems out of hand, so we consider some special cases.

When Does UDP = UDA?

There are cases when UDP $\neq$ UDA (e.g., the $d = 3$ case of 7 observables).

When can we guarantee that UDP = UDA (i.e., when can we guarantee that $|\phi\rangle$ being UDP implies it is UDA)?

A complete characterization seems out of hand, so we consider some special cases.

When Does UDP = UDA?

There are cases when UDP $\neq$ UDA (e.g., the $d = 3$ case of 7 observables).

When can we guarantee that UDP = UDA (i.e., when can we guarantee that $|\phi\rangle$ being UDP implies it is UDA)?

A complete characterization seems out of hand, so we consider some special cases.

The One- and Two-Observable Case

Suppose that there are just two observables. There should not be many pure states $|\phi\rangle$ that are UDA, or even UDP.

However, it can happen. For example, if A_1 has a distinct minimal (or maximal) eigenvalue λ, then the corresponding eigenvector $|\phi\rangle$ is UDA (and hence also UDP):

$$\mathrm{Tr}(A_1\rho) = \lambda \iff \rho = |\phi\rangle\langle\phi|.$$

Thus, it's possible that some pure states are UDA even in the one-observable case.

The One- and Two-Observable Case

Suppose that there are just two observables. There should not be many pure states $|\phi\rangle$ that are UDA, or even UDP.

However, it **can** happen. For example, if A_1 has a distinct minimal (or maximal) eigenvalue λ, then the corresponding eigenvector $|\phi\rangle$ is UDA (and hence also UDP):

$$\mathrm{Tr}(A_1 \rho) = \lambda \iff \rho = |\phi\rangle\langle\phi|.$$

Thus, it's possible that some pure states are UDA even in the one-observable case.

The One- and Two-Observable Case

Suppose that there are just two observables. There should not be many pure states $|\phi\rangle$ that are UDA, or even UDP.

However, it **can** happen. For example, if A_1 has a distinct minimal (or maximal) eigenvalue λ, then the corresponding eigenvector $|\phi\rangle$ is UDA (and hence also UDP):

$$\mathrm{Tr}(A_1\rho) = \lambda \Longleftrightarrow \rho = |\phi\rangle\langle\phi|.$$

Thus, it's possible that some pure states are UDA even in the one-observable case.

The One- and Two-Observable Case

In these cases, UDP = UDA always:

Theorem

In the case of one or two observables (i.e., $\mathbf{A} = (A_1, A_2)$), UDP = UDA for all pure states $|\phi\rangle$.

The One- and Two-Observable Case

In these cases, UDP = UDA always:

Theorem

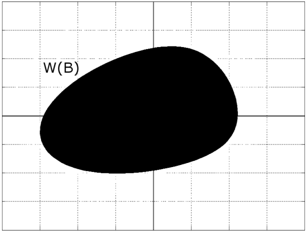
In the case of one or two observables (i.e., $\mathbf{A} = (A_1, A_2)$), UDP = UDA for all pure states $|\phi\rangle$.

Rough Idea of Proof

Recall that the numerical range of an operator B is the following set of complex numbers:

$$W(B) := \big\{\langle\phi|B|\phi\rangle : |\phi\rangle \in \mathcal{H}_d\big\}.$$

We can always write $B = A_1 + iA_2$ for some Hermitian A_1, A_2, so the set of possible pure state measurement outcomes is just the numerical range $W(B)$.

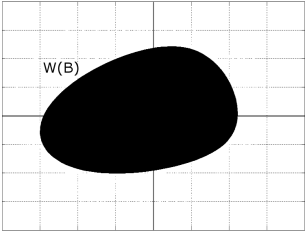


Rough Idea of Proof

Recall that the numerical range of an operator B is the following set of complex numbers:

$$W(B) := \big\{\langle\phi|B|\phi\rangle : |\phi\rangle \in \mathcal{H}_d\big\}.$$

We can always write $B = A_1 + iA_2$ for some Hermitian A_1, A_2, so the set of possible pure state measurement outcomes is just the numerical range $W(B)$.

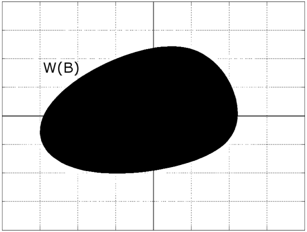


Rough Idea of Proof

Recall that the numerical range of an operator B is the following set of complex numbers:

$$W(B) := \left\{ \langle\phi|B|\phi\rangle : |\phi\rangle \in \mathcal{H}_d \right\}.$$

The Hausdorff–Toeplitz theorem says that $W(B)$ is convex, so this also equals the set of all possible mixed state measurement outcomes.



Rough Idea of Proof

The result now follows by combining the following two (fairly straightforward) facts:

Fact 1: If $|\phi\rangle$ is UDP, then $\langle\phi|B|\phi\rangle$ is an extreme point of $W(B)$.

Fact 2: If $\mathbf{A}(\rho)$ is an extreme point of $W(B)$, then $\mathbf{A}(\rho) = \mathbf{A}(|\phi\rangle)$ for all $|\phi\rangle$ in the range of ρ.

Rough Idea of Proof

The result now follows by combining the following two (fairly straightforward) facts:

Fact 1: If $|\phi\rangle$ is UDP, then $\langle\phi|B|\phi\rangle$ is an extreme point of $W(B)$.

Fact 2: If $\mathbf{A}(\rho)$ is an extreme point of $W(B)$, then $\mathbf{A}(\rho) = \mathbf{A}(|\phi\rangle)$ for all $|\phi\rangle$ in the range of ρ.

Rough Idea of Proof

The result now follows by combining the following two (fairly straightforward) facts:

Fact 1: If $|\phi\rangle$ is UDP, then $\langle\phi|B|\phi\rangle$ is an extreme point of $W(B)$.

Fact 2: If $\mathbf{A}(\rho)$ is an extreme point of $W(B)$, then $\mathbf{A}(\rho) = \mathbf{A}(|\phi\rangle)$ for all $|\phi\rangle$ in the range of ρ.

The Two-Dimensional Case

The previous theorem leads easily to the following result, which solves the two-dimensional case:

Corollary

When $d = 2$, UDP = UDA for all pure states $|\phi\rangle$.

Proof: The previous result demonstrates this when there is only 1 or 2 observables. When there are 3 observables, every pure state is UDA (and hence UDP).

The Two-Dimensional Case

The previous theorem leads easily to the following result, which solves the two-dimensional case:

Corollary

When $d = 2$, UDP = UDA for all pure states $|\phi\rangle$.

Proof: The previous result demonstrates this when there is only 1 or 2 observables. When there are 3 observables, every pure state is UDA (and hence UDP).

The Two-Dimensional Case

The previous theorem leads easily to the following result, which solves the two-dimensional case:

Corollary

When $d = 2$, UDP = UDA for all pure states $|\phi\rangle$.

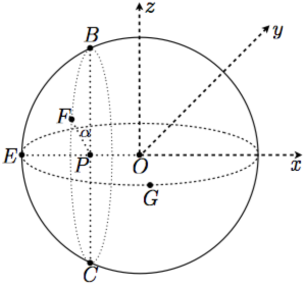
Proof: The previous result demonstrates this when there is only 1 or 2 observables. When there are 3 observables, every pure state is UDA (and hence UDP).

Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Let K_d be the d-dimensional state space (i.e., the $d \times d$ density matrices).

When $d = 2$, K_d is the Bloch ball (shown to the right).

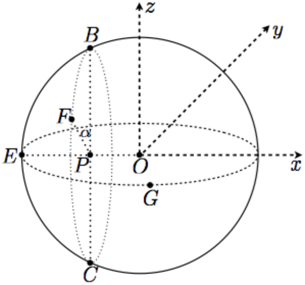


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Let K_d be the d-dimensional state space (i.e., the $d \times d$ density matrices).

When $d = 2$, K_d is the Bloch ball (shown to the right).

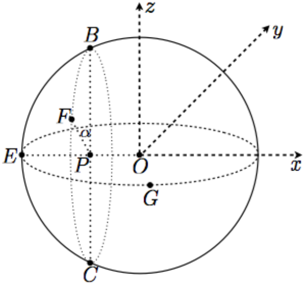


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Let K_d be the d-dimensional state space (i.e., the $d \times d$ density matrices).

When $d = 2$, K_d is the Bloch ball (shown to the right).

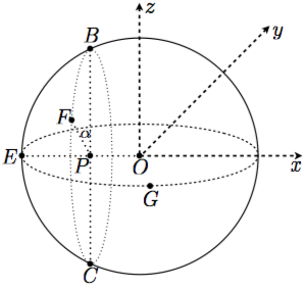


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Measuring the Pauli X and Y operators projects the Bloch ball down to the xy-plane. The Bloch sphere is symmetric with respect to reflection across the xy-plane.

Mirrored pure states give the same measurement results (e.g., B and C), as does any state on the line between them.

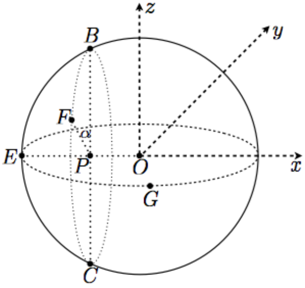


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Measuring the Pauli X and Y operators projects the Bloch ball down to the xy-plane. The Bloch sphere is symmetric with respect to reflection across the xy-plane.

Mirrored pure states give the same measurement results (e.g., B and C), as does any state on the line between them.

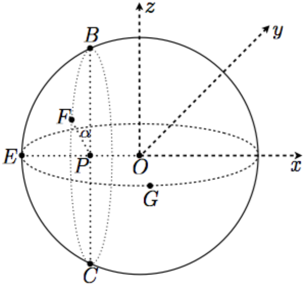


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Thus either a pure state is neither UDP nor UDA (e.g., points B and C), or...

A point is both UDP and UDA (i.e., the states on the equator, since they are fixed by the projection onto the xy-plane).

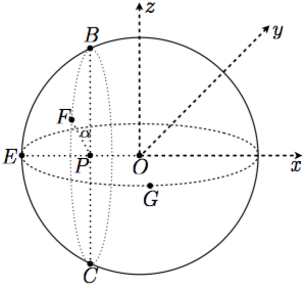


Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Thus either a pure state is neither UDP nor UDA (e.g., points B and C), or...

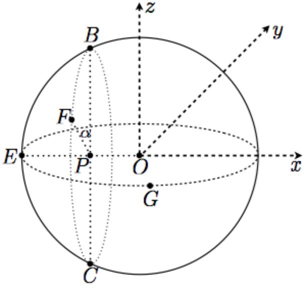
A point is both UDP and UDA (i.e., the states on the equator, since they are fixed by the projection onto the xy-plane).



Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Measuring just the Pauli-X operator projects the Bloch ball down to the x-axis. The Bloch sphere is symmetric with respect to rotation about the x-axis.



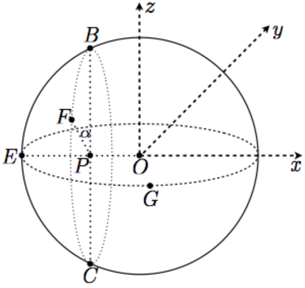
Thus the only two points that are UDP are the left and right poles. They are also UDA.

Symmetry of the State Space

To motivate another case where UDP = UDA, we consider a 2-dimensional example.

Measuring just the Pauli-X operator projects the Bloch ball down to the x-axis. The Bloch sphere is symmetric with respect to rotation about the x-axis.

Thus the only two points that are UDP are the left and right poles. They are also UDA.



Symmetry of the State Space

The previous observations generalize naturally, and lead to the following result:

Theorem

If there is a compact group of affine automorphisms of K_d whose fixed point set is $K_d \bigcap \mathrm{span}(\mathbf{A})$, then UDP = UDA for $\mathbf{A}$.

Reduced Density Matrices

We can also ask whether a pure state is UDP and/or UDA by some of its reduced density matrices.

For example, can we uniquely determine $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ from $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$?

To put this into the framework we've been using, we can let $\mathbf{A}$ be a complete family of observables acting on the systems that are not being traced out.

In this case (if $d_1 = 2$), knowing $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ corresponds to measuring $\mathbf{A} = \{X \otimes I, Y \otimes I, Z \otimes I\}$.

Reduced Density Matrices

We can also ask whether a pure state is UDP and/or UDA by some of its reduced density matrices.

For example, can we uniquely determine $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ from $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$?

To put this into the framework we've been using, we can let $\mathbf{A}$ be a complete family of observables acting on the systems that are not being traced out.

In this case (if $d_1 = 2$), knowing $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ corresponds to measuring $\mathbf{A} = \{X \otimes I, Y \otimes I, Z \otimes I\}$.

Reduced Density Matrices

We can also ask whether a pure state is UDP and/or UDA by some of its reduced density matrices.

For example, can we uniquely determine $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ from $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$?

To put this into the framework we've been using, we can let $\mathbf{A}$ be a complete family of observables acting on the systems that are not being traced out.

In this case (if $d_1 = 2$), knowing $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ corresponds to measuring $\mathbf{A} = \{X \otimes I, Y \otimes I, Z \otimes I\}$.

Reduced Density Matrices

We can also ask whether a pure state is UDP and/or UDA by some of its reduced density matrices.

For example, can we uniquely determine $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ from $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$?

To put this into the framework we've been using, we can let **A** be a complete family of observables acting on the systems that are not being traced out.

In this case (if $d_1 = 2$), knowing $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ corresponds to measuring $\mathbf{A} = \{X \otimes I, Y \otimes I, Z \otimes I\}$.

Some Results on Reduced Density Matrices

Some results in the tripartite case (i.e., the case of $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$) are:

If $d_1 \geq 2$ then almost every pure state $|\phi\rangle$ is UDP by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$ [3].

If $d_1 \geq \min(d_2, d_3)$ then almost every pure state $|\phi\rangle$ is UDA by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$.

[3] *L. Diósi. Three-party pure quantum states are determined by two two-party reduced states. Phys. Rev. A, 70:010302, 2004.*

Some Results on Reduced Density Matrices

Some results in the tripartite case (i.e., the case of $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$) are:

If $d_1 \geq 2$ then almost every pure state $|\phi\rangle$ is UDP by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$ [3].

If $d_1 \geq \min(d_2, d_3)$ then almost every pure state $|\phi\rangle$ is UDA by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$.

[3] *L. Diósi. Three-party pure quantum states are determined by two two-party reduced states. Phys. Rev. A, 70:010302, 2004.*

Some Results on Reduced Density Matrices

Some results in the tripartite case (i.e., the case of $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$) are:

If $d_1 \geq 2$ then almost every pure state $|\phi\rangle$ is UDP by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$ [3].

If $d_1 \geq \min(d_2, d_3)$ then almost every pure state $|\phi\rangle$ is UDA by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$.

[3] *L. Diósi. Three-party pure quantum states are determined by two two-party reduced states. Phys. Rev. A, 70:010302, 2004.*

Closing Questions

How large can subspaces of Hermitian matrices with bounded rank (or eigenvalue signs) be?

What are some other cases where UDP = UDA?

Can we close the gap in the tripartite reduced density matrix case?

Thank You!

J. Chen, H. Dawkins, Z. Ji, N. Johnston, D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. arXiv:1212.3503 [quant-ph], 2012.

Closing Questions

How large can subspaces of Hermitian matrices with bounded rank (or eigenvalue signs) be?

What are some other cases where UDP = UDA?

Can we close the gap in the tripartite reduced density matrix case?

Thank You!

J. Chen, H. Dawkins, Z. Ji, N. Johnston, D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. arXiv:1212.3503 [quant-ph], 2012.

Closing Questions

How large can subspaces of Hermitian matrices with bounded rank (or eigenvalue signs) be?

What are some other cases where UDP = UDA?

Can we close the gap in the tripartite reduced density matrix case?

Thank You!

J. Chen, H. Dawkins, Z. Ji, N. Johnston, D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. arXiv:1212.3503 [quant-ph], 2012.

Closing Questions

How large can subspaces of Hermitian matrices with bounded rank (or eigenvalue signs) be?

What are some other cases where UDP = UDA?

Can we close the gap in the tripartite reduced density matrix case?

Thank You!

J. Chen, H. Dawkins, Z. Ji, N. Johnston, D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. arXiv:1212.3503 [quant-ph], 2012.