Uniqueness of Quantum States Compatible with Given Measurement Results

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Mathematical Setup

We have a tuple of m observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the pure state $|\phi\rangle$, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(|\phi\rangle) := (\langle\phi|A_1|\phi\rangle, \ldots, \langle\phi|A_m|\phi\rangle).$$

We want $|\phi\rangle$ to be uniquely determined by $\mathbf{A}(|\phi\rangle)$.

The Main Question

Is a given pure state $|\phi\rangle$ uniquely determined under measuring a set of observables $\{A_1, \ldots, A_m\}$?

What Do We Mean?

When we ask that $|\phi\rangle$ is uniquely determined by its measurement results, what exactly do we mean?

- No other pure state has the same measurement results?
- No other state (pure or mixed) has the same measurement results?

Ideal: We know that our state is pure.

Real world: Maybe it got mixed.

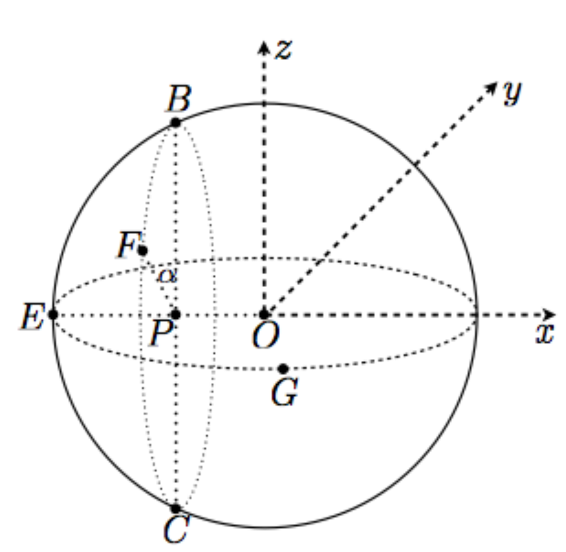

Figure 1: The Bloch ball, which is useful for visualizing the two types of uniqueness.

Two Types of Uniqueness

A pure state is **uniquely determined among pure states (UDP)** for $\mathbf{A}$ if there does not exist any other pure state with the same measurement results.

A pure state is **uniquely determined among all states (UDA)** for $\mathbf{A}$ if there does not exist any other state (pure or mixed) with the same measurement results.

Fewest Observables for UDP

How few measurement results suffice so that every pure state is UDP?

- Trivially, d^2-1 measurement results suffice, as the space spanned by all density matrices is (d^2-1)-dimensional.
- It is known [1] that there exists a family of $m = 4d-5$ observables so that every pure state is UDP.
- However, $4d-5$ isn't quite optimal (the optimal value is not known for $d \geq 8$).
- Equivalent to asking what the largest (real) subspace of Hermitian matrices of rank ≥ 3 is (the complex version of this problem was answered in [2]).

Fewest Observables for UDA

How few measurement results suffice so that every pure state is UDA?

- There exists a family of $m = 5d-7$ observables so that every pure state is UDA.
- It is unknown whether or not $5d-7$ is optimal.
- Equivalent to asking what the largest (real) subspace of Hermitian matrices each with at least 2 positive and 2 negative eigenvalues is ($5d-7 \iff$ subspace of dimension $(d-2)(d-3)$).

When Does UDP = UDA?

It is clear that if a pure state is UDA by its measurement results, then it is UDP as well. When does the converse hold?

Example 1: One or Two Observables

- Intuitively, there should not be many pure states $|\phi\rangle$ that are UDA, or even UDP.
- However, it *can* happen (e.g., if A_1 has a distinct minimal eigenvalue, then the corresponding eigenvector is UDP and UDA).

Theorem. In the case of one or two observables (i.e., $m \leq 2$), UDP = UDA for all pure states $|\phi\rangle$.

Example 2: Symmetry of the State Space

- Let K_d be the set of $d \times d$ density matrices.
- Measuring the Pauli X and Y operators projects the Bloch ball (see Figure 1) down to the xy-plane.
- Mirrored pure states give the same measurement results (e.g., B and C).
- Either a pure state is neither UDP nor UDA (e.g., points B and C), or a point is both UDP and UDA (e.g., points E and G).

Theorem. If there is a compact group of affine automorphisms of K_d whose fixed point set is $K_d \cap \mathrm{span}(\mathbf{A})$, then UDP = UDA.

Reduced Density Matrices

When is a pure state UDP and/or UDA by some of its reduced density matrices? There are some results in the tripartite case (i.e., $|\phi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$):

- If $d_1 \geq 2$ then almost every pure state $|\phi\rangle$ is UDP by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$ [3].
- If $d_1 \geq \min(d_2, d_3)$ then almost every pure state $|\phi\rangle$ is UDA by $\mathrm{Tr}_2(|\phi\rangle\langle\phi|)$ and $\mathrm{Tr}_3(|\phi\rangle\langle\phi|)$.

For Further Information

For the details of our work, see:

- J. Chen, H. Dawkins, Z. Ji, N. Johnston, D. W. Kribs, F. Shultz, and B. Zeng. *Uniqueness of quantum states compatible with given measurement results.* E-print: arXiv:1212.3503 [quant-ph], 2012.

References

[1] T. Heinosaari, L. Mazzarella, and M. M. Wolf. *Quantum tomography under prior information.* E-print: arXiv:1109.5478 [quant-ph], 2011.

[2] T. S. Cubitt, A. Montanaro, and A. Winter. *On the dimension of subspaces with bounded Schmidt rank.* J. Math. Phys., 49:022107, 2008.

[3] L. Diósi. *Three-party pure quantum states are determined by two two-party reduced states.* Phys. Rev. A, 70:010302, 2004.

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