Non-Uniqueness of Minimal Superpermutations

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De Bruijn Sequences

Consider a string on n symbols that has every word of length k as a substring.

Example ($n = k = 2$): The string 11221 has the desired property, since it contains each of 11, 12, 21, and 22 as a substring:

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How small can strings with this property be? A shortest string with this property is called a **de Bruijn sequence**.

A trivial lower bound is $n^k + k - 1$, since there are n^k words of length k on n symbols, and any string with length L has $L - k + 1$ substrings of length k.

More interestingly, this lower bound is attained – de Bruijn sequences have length $n^k + k - 1$.

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Superpermutations

A **superpermutation** is a string on n symbols that contains every permutation of those n symbols as a substring.

Example (n = 3): The string 123121321 is a superpermutation on the three symbols 1, 2, and 3, since it contains each of 123, 132, 213, 231, 312, and 321 as substrings:

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123121321	12312**132**1
1231**213**21	1**231**21321
12**312**1321	123121**321**

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How small can strings with this property be? A shortest string with this property is called a **minimal superpermutation**.

A trivial lower bound is $n! + n - 1$, since there are $n!$ permutations of n symbols, and any string with length L has $L - n + 1$ substrings of length n (does this seem familiar?).

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Superpermutations

For example, when $n = 3$, minimal superpermutations have length 9, not $3! + 3 - 1 = 8$.

- We already saw a superpermutation of length 9: 123121321
- Use computer search to see that no superpermutation of length 8 exists.

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Length when $n \leq 4$

When $n \leq 4$, the length of minimal superpermutations can be found via computer search:

n	Minimal Superpermutation	Length
1	1	$1 = 1!$
2	121	$3 = 1! + 2!$
3	123121321	$9 = 1! + 2! + 3!$
4	123412314231243121342132413214321	$33 = 1! + 2! + 3! + 4!$

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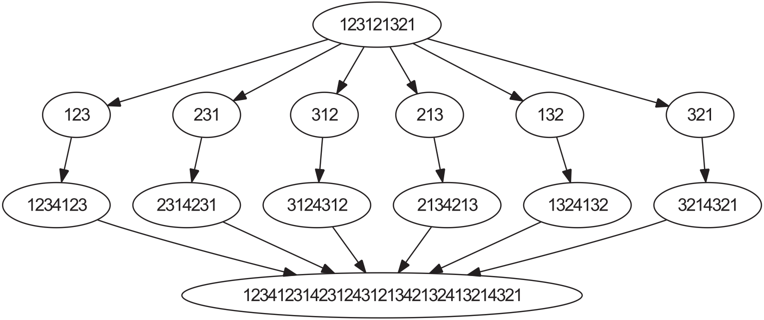
General Construction

Nonetheless, there is a simple recursive construction that produces small superpermutations for any n:



General Construction

It is straightforward to show that:

- The superpermutations produced in this way have length $\sum_{k=1}^{n} k!$
- This procedure generates the superpermutations given in the earlier table when $n \leq 4$.
- When $n \leq 4$, these superpermutation are minimal, and furthermore they are unique (up to requiring that they start with $12 \cdots n$).

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The Conjecture

These observations have led to the following conjecture [1]:

Conjecture

For all $n \geq 1$, the minimal superpermutation on n symbols has length $\sum_{k=1}^{n} k!$ and is unique.

[1] D. Ashlock and J. Tillotson. Construction of small superpermutations and minimal injective superstrings. *Congressus Numerantium*, 93:91–98, 1993.

Non-Uniqueness when $n = 5$

It turns out that uniqueness is false. In the $n = 5$ case, there are at least 2 superpermutations of the conjectured minimal length 153:

123451234152341253412354123145231425314235142315423·

·1245312435124315243125431**2**1345213425134215342135421·

·324513241532413524132541321453214352143251432154321

and

123451234152341253412354123145231425314235142315423·

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Non-Uniqueness: Main Result

More generally, we have shown that that uniqueness conjecture is **extremely** false. Our main result is:

Theorem

There are at least $\prod_{k=1}^{n-4}(n-k-2)!^{k \cdot k!}$ distinct superpermutations on n symbols of the conjectured minimum length.

For $n \leq 4$, this formula equals 1, which agrees with uniqueness in these cases.

For $n = 5, 6, 7, 8$, this formula gives the values $2, 96, 8153726976$, and approximately 3×10^{50}, respectively.

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Open Questions

Are there superpermutations of length less than $\sum_{k=1}^{n} n!$ when $n \geq 5$?

We have shown that there are many superpermutations of length $\sum_{k=1}^{n} n!$. Have we found them all or are there even more?

N. Johnston. Non-uniqueness of minimal superpermutations. *Discrete Mathematics*, 313:1553–1557, 2013. arXiv:1303.4150 [math.CO]

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