Norm Duality and the Cross Norm Criteria for Quantum Entanglement

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Abstract

We use duality techniques to prove and generalize the cross norm and computable cross norm criteria for separability of quantum states. While the original proof of the cross norm criterion is long and involved, our new proof is short and elementary. Furthermore, our proof generalizes naturally to arbitrary Schmidt number. We also use these techniques to generalize the computable cross norm criterion to arbitrary Schmidt number and prove some results of independent interest along the way.

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1. Introduction

In quantum information theory, the distinction between separable states and entangled states is one of the most fundamental and important concepts. However, the problem of determining whether or not a given quantum state is entangled, even given a complete mathematical representation of the state in question, is very difficult and an active area of research. Some of the more well-known tests for detecting entanglement include the *cross norm criterion* [Rud00] (which is very strong, but difficult to apply in practice) and the *computable cross norm criterion* [CW03, Rud03] (which is weaker, but easy to apply in practice).

A notion that generalizes that of separability is that of *Schmidt number* [TH00], which is a positive integer that provides a rough measure of "how entangled" a quantum state is. A state is separable if and only if its Schmidt number is 1, and higher Schmidt numbers indicate more entanglement. The goal of the present paper is to generalize the cross norm criterion and the computable cross norm criterion to arbitrary Schmidt number, and to do so in a way that is reasonably elementary compared to existing proofs.

Our approach is to consider norm duality. After introducing the basics of quantum entanglement in Section 2, we prove a general norm duality result that is of independant

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interest in Section 3. To demonstrate the power of this duality result, we briefly discuss some of its consequences, including a characterization of norms that are invariant under the action of local unitary operators. We then apply our duality result to the $S(k)$-*norm* [JK10, JK11] from quantum information theory, and we see that its dual is the *projective tensor norm* [Rud00, Rud05] in Section 4. We then show that this result easily gives the desired generalization of the cross norm criterion.

Finally, we close in Section 5 by using similar duality techniques to generalize the computable cross norm criterion. We show that, while the standard computable cross norm criterion depends on the trace norm, its natural generalization depends on the dual of a norm that can be thought of as a mix between the Frobenius and Ky Fan k-norms. We show that our criterion is both necessary and sufficient on pure states, and we numerically analyze its effectiveness in general.

2. Notation and Preliminaries

Here we introduce our notation and terminology. We use $\mathcal{H}$ to denote a finite-dimensional complex Hilbert space. We typically represent vectors $\mathbf{v} \in \mathcal{H}$ using boldface, but if we wish to emphasize that the vector in question has unit length (with respect to the norm induced by the inner product), then we use "ket" notation: $|v\rangle \in \mathcal{H}$. In this case, we use "bras" to represent dual (i.e., row) vectors: $\langle v| := |v\rangle^*$.

The Hilbert spaces of primary interest to us are $\mathbb{C}^n$ (i.e., n-dimensional complex Euclidean space) and M_n, the space of $n \times n$ complex matrices equipped with the Hilbert–Schmidt inner product $\langle A, B\rangle := \mathrm{Tr}(A^*B)$. We also consider $M_m \otimes M_n$ as a Hilbert space with the same inner product, but when possible we don't specify a particular Hilbert space at all for the sake of generality.

A pure quantum state is represented by a unit vector $|v\rangle \in \mathbb{C}^n$. A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called *separable* if it can be written in the form $|v\rangle = |a\rangle \otimes |b\rangle$ for some $|a\rangle \in \mathbb{C}^m$, $|b\rangle \in \mathbb{C}^n$, and it is called *entangled* otherwise. The *Schmidt rank* of a pure state $|v\rangle$, which we denote by $SR(|v\rangle)$, is the least integer k so that we can write $|v\rangle = \sum_{i=1}^k c_i|v_i\rangle$ with each $|v_i\rangle$ separable. It is the case that $1 \leq SR(|v\rangle) \leq \min\{m, n\}$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ and $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.

While pure states are simple to work with mathematically, not all quantum states are pure. General (i.e., potentially *mixed*) quantum states are represented by *density matrices*: positive semidefinite matrices $\rho = \rho^* \in M_n$ satisfying $\mathrm{Tr}(\rho) = 1$. If $|v\rangle$ represents a pure state then the projection onto its span, $|v\rangle\langle v|$, is its density matrix representation. A general density matrix ρ can be written as a convex combination of pure states: $\rho = \sum_i p_i|v_i\rangle\langle v_i|$ with $\sum_i p_i = 1$ and $p_i \geq 0$ for all i. If ρ can be written in this way with each $|v_i\rangle$ separable then we say that ρ is *separable* [Wer89]. More generally, the *Schmidt number* of ρ, denoted $SN(\rho)$, is the least integer k such that ρ can be written in this form with each $|v_i\rangle$ having $SR(|v_i\rangle) \leq k$ [TH00]. If $SN(\rho) \geq 2$ then ρ is called *entangled*.

An operator $Y = Y^* \in M_m \otimes M_n$ is called *k-block positive* if $\langle v|Y|v\rangle \geq 0$ whenever $SR(|v\rangle) \leq k$. The sets of k-block positive operators and states with Schmidt number at most k are dual to each other in the sense that $SN(\rho) \leq k$ if and only if $\mathrm{Tr}(\rho Y) \geq 0$ for all k-block positive Y [SSŻ09].

If we fix $1 \leq k \leq \min\{m,n\}$ then two norms on $M_m \otimes M_n$ that play a key role in the remainder of this paper are as follows:

$$\big\|X\big\|_{S(k)} := \sup\Big\{\big|\langle v|X|w\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k\Big\} \text{ and} \tag{1}$$

$$\big\|X\big\|_{\gamma,k} := \inf\Big\{\sum_i |c_i| : X = \sum_i c_i|v_i\rangle\langle w_i| \text{ with } SR(|v_i\rangle), SR(|w_i\rangle) \leq k \;\forall\, i\Big\}, \tag{2}$$

where the supremum (1) is taken over all $|v\rangle, |w\rangle$ satisfying the Schmidt rank condition and the infimum (2) is taken over all decompositions of the indicated form.

Notice that in the $k = \min\{m,n\}$ case we have

$$\big\|X\big\|_{S(\min\{m,n\})} = \big\|X\big\| \quad \text{and} \quad \big\|X\big\|_{\gamma,\min\{m,n\}} = \big\|X\big\|_{tr},$$

where $\|\cdot\|$ and $\|\cdot\|_{tr}$ refer to the *operator norm* and *trace norm*, respectively, defined as follows:

$$\big\|X\big\| := \sup\Big\{\big|\langle v|X|w\rangle\big|\Big\} \quad \text{and} \quad \big\|X\big\|_{tr} := \sup\Big\{\big|\mathrm{Tr}(XU)\big| : \; U \in M_n \text{ is unitary}\Big\}.$$

The norm (1) was introduced in [JK10, JK11] as a tool for investigating k-block positivity. In particular, we have the following simple proposition, which was originally proved as [JK10, Corollary 4.9], but we reprove here for completeness.

Proposition 1. *Let $Y = Y^* \in M_m \otimes M_n$. If we write $Y = cI - X$ with X positive semidefinite, then Y is k-block positive if and only if $c \geq \|X\|_{S(k)}$.*

Proof. Let $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ have $SR(|v\rangle) \leq k$. Then

$$\langle v|Y|v\rangle = \langle v|(cI - X)|v\rangle = c - \langle v|X|v\rangle. \tag{3}$$

By taking the infimum of Equation (3) over all such $|v\rangle$ (and using the easily-verified fact that the supremum defining the norm (1) is attained when $|v\rangle = |w\rangle$ when X is positive semidefinite), the result follows. □

On the other hand, the norm (2), in the $k = 1$ case, was studied in relation to quantum entanglement in [Rud00, Rud05] and is called the *projective tensor norm*. Observe in this case that it can be written in the following slightly simpler form:

$$\begin{aligned}\big\|X\big\|_{\gamma,1} &= \inf\Big\{\sum_i |c_i| : X = \sum_i c_i|v_i\rangle\langle w_i| \otimes |x_i\rangle\langle y_i|\Big\} \\ &= \inf\Big\{\sum_i \big\|A_i\big\|_{tr}\big\|B_i\big\|_{tr} : X = \sum_i A_i \otimes B_i\Big\}.\end{aligned}$$

Much like $\|\cdot\|_{S(k)}$ characterizes k-block positivity, it is known that $\|\cdot\|_{\gamma,1}$ characterizes separability in the sense that a density matrix ρ is separable if and only if $\|\rho\|_{\gamma,1} = 1$. However, in contrast with the simplicity of the proof of Proposition 1, the proof of this fact is quite long and complicated. In the next section we introduce norm duality, and show that it leads to an elementary new proof of (a generalization of) this result.

3. Dual Norms

Given a norm $|||\cdot|||$ on $\mathcal{H}$ (not necessarily equal to the norm induced by the inner product), the *dual norm* of $|||\cdot|||$ is defined by

$$|||\mathbf{v}|||^\circ := \sup\Big\{\big|\langle \mathbf{w}, \mathbf{v}\rangle\big| : |||\mathbf{w}||| \leq 1\Big\}. \tag{4}$$

For example, when $\mathcal{H} = M_n$, the *Frobenius norm* is the norm induced by the inner product and is thus self-dual: $\|X\|_F := \sqrt{\mathrm{Tr}(X^*X)} = \|X\|_F^\circ$. As a less trivial example, it is well-known that the operator norm and the trace norm are dual to each other: $\|\cdot\|^\circ = \|\cdot\|_{tr}$.

The following result allows us to rephrase dual norms, which so far we have written as the supremum (4), as an infimum. We expect that this result is known, though we have not been able to find a reference for it.

Theorem 2. *Let $\mathcal{S} \subseteq \mathcal{H}$ be a bounded set satisfying* $\mathrm{span}(\mathcal{S}) = \mathcal{H}$ *and define a norm $|||\cdot|||$ by*

$$|||\mathbf{v}||| := \sup_{\mathbf{w}\in\mathcal{S}}\Big\{\big|\langle \mathbf{v}, \mathbf{w}\rangle\big|\Big\}.$$

Then $|||\cdot|||^\circ$ is given by

$$|||\mathbf{v}|||^\circ = \inf\Big\{\sum_i |c_i| : \mathbf{v} = \sum_i c_i\mathbf{v}_i, \text{ where } c_i \in \mathbb{F}, \mathbf{v}_i \in \mathcal{S}\ \forall\, i\Big\},$$

where the infimum is taken over all such decompositions of $\mathbf{v}$.

We make some observations before proving the result. Firstly, the conditions placed on $\mathcal{S}$ by Theorem 2 are both necessary and sufficient for the quantity $|||\cdot|||$ to be a norm: boundedness of $\mathcal{S}$ ensures that the supremum is finite, and $\mathrm{span}(\mathcal{S}) = \mathcal{H}$ is equivalent to the statement that $|||\mathbf{v}||| = 0$ if and only if $\mathbf{v} = 0$. Secondly, every norm on $\mathcal{H}$ can be written in this form: we can always choose $\mathcal{S}$ to be the unit ball of the dual norm $|||\cdot|||^\circ$. However, it is sometimes useful to make other choices of $\mathcal{S}$ such as the set of extreme points of the unit ball of the dual norm.

Proof of Theorem 2. Begin by noting that if $\mathbf{w} \in \mathcal{S}$ and $|||\mathbf{v}||| \leq 1$ then $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq 1$. It follows that $|||\mathbf{w}|||^\circ \leq 1$ whenever $\mathbf{w} \in \mathcal{S}$. In fact, we now show that $|||\cdot|||^\circ$ is the largest norm on $\mathcal{H}$ with this property. To this end, let $|||\cdot|||_2$ be another norm satisfying $|||\mathbf{w}|||_2^\circ \leq 1$ whenever $\mathbf{w} \in \mathcal{S}$. Then

$$|||\mathbf{v}||| = \sup_{\mathbf{w}\in\mathcal{S}}\Big\{\big|\langle \mathbf{v}, \mathbf{w}\rangle\big|\Big\} \leq \sup_{\mathbf{w}}\Big\{\big|\langle \mathbf{v}, \mathbf{w}\rangle\big| : |||\mathbf{w}|||_2^\circ \leq 1\Big\} = |||\mathbf{v}|||_2\,.$$

Thus $|||\cdot||| \leq |||\cdot|||_2$, so by taking duals we see that $|||\cdot|||^\circ \geq |||\cdot|||_2^\circ$, as desired.

For the remainder of the proof, we denote the infimum in the statement of the theorem by $\|\cdot\|_{\mathrm{inf}}$. Our goal now is to show that: (a) $\|\cdot\|_{\mathrm{inf}}$ is a norm, (b) $\|\cdot\|_{\mathrm{inf}}$ satisfies $\|\mathbf{w}\|_{\mathrm{inf}} \leq 1$ whenever $\mathbf{w} \in \mathcal{S}$, and (c) $\|\cdot\|_{\mathrm{inf}}$ is the largest norm satisfying property (b). The fact that

$\|\cdot\|_{\inf} = |||\cdot|||^\circ$ will then follow from the fact that $|||\cdot|||^\circ$ is also the largest norm satisfying property (b).

To see (a) (i.e., to prove that $\|\cdot\|_{\inf}$ is a norm), we only prove the triangle inequality, since the other properties are trivial. Fix $\varepsilon > 0$ and let $\mathbf{v} = \sum_i c_i \mathbf{v}_i$, $\mathbf{w} = \sum_i d_i \mathbf{w}_i$ be decompositions of $\mathbf{v}, \mathbf{w}$ with $\mathbf{v}_i, \mathbf{w}_i \in \mathcal{S}$ for all i, satisfying $\sum_i |c_i| \leq \|\mathbf{v}\|_{\inf} + \varepsilon$ and $\sum_i |d_i| \leq \|\mathbf{w}\|_{\inf} + \varepsilon$. Then

$$\|\mathbf{v} + \mathbf{w}\|_{\inf} \leq \sum_i |c_i| + \sum_i |d_i| \leq \|\mathbf{v}\|_{\inf} + \|\mathbf{w}\|_{\inf} + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the triangle inequality follows, so $\|\cdot\|_{\inf}$ is a norm.

To see (b) (i.e., to prove that $\|\mathbf{v}\|_{\inf} \leq 1$ whenever $\mathbf{v} \in \mathcal{S}$), we simply write $\mathbf{v}$ in its trivial decomposition $\mathbf{v} = \mathbf{v}$, which gives $\|\mathbf{v}\|_{\inf} \leq \sum_i c_i = c_1 = 1$.

To see (c) (i.e., to prove that $\|\cdot\|_{\inf}$ is the largest norm on $\mathcal{H}$ satisfying condition (b)), begin by letting $|||\cdot|||_2$ be any norm on $\mathcal{H}$ with the property that $|||\mathbf{v}|||_2 \leq 1$ for all $\mathbf{v} \in \mathcal{S}$. Then using the triangle inequality for $|||\cdot|||_2$ shows that if $\mathbf{v} = \sum_i c_i \mathbf{v}_i$ is any decomposition of $\mathbf{v}$ with $\mathbf{v}_i \in \mathcal{S}$ for all i, then

$$|||\mathbf{v}|||_2 = \left|\left|\left|\sum_i c_i \mathbf{v}_i\right|\right|\right|_2 \leq \sum_i |c_i| \, |||\mathbf{v}_i|||_2 \leq \sum_i |c_i|.$$

Taking the infimum over all such decompositions of $\mathbf{v}$ shows that $|||\mathbf{v}|||_2 \leq \|\mathbf{v}\|_{\inf}$, which completes the proof. □

As an example of an application of Theorem 2, we again consider the operator norm and trace norm on M_n, which we already noted are dual to each other. The theorem then says that

$$\big\|X\big\| = \inf\Big\{\sum_i |c_i| : X = \sum_i c_i U_i \text{ with each } U_i \text{ unitary}\Big\}, \text{ and} \tag{5}$$

$$\big\|X\big\|_{tr} = \inf\Big\{\sum_i |c_i| : X = \sum_i c_i |w_i\rangle\langle v_i|\Big\}. \tag{6}$$

The above characterization of $\|\cdot\|_{tr}$ is well-known, and the infimum is attained when we write X in its singular value decomposition. The characterization of $\|\cdot\|$ is perhaps slightly less well-known and interesting in its own right. Theorem 2 also generalizes the fact that the injective and projective tensor norms are dual to each other (see [DFS08, Chapter 1]) and the fact that the 1-norm and ∞-norm on $\mathbb{C}^n$ are dual to each other.

As an application of the formula (5) for the operator norm, recall [Bha97, Proposition IV.2.4], which says that a norm $|||\cdot|||$ is unitarily-invariant if and only if $|||AXB||| \leq \|A\|\,|||X|||\,\|B\|$ for all $A, X, B \in M_n$. The standard proof of this fact uses singular values and thus does not generalize in any natural way to tensor product systems. However, Equation (5) allows us to easily prove the following natural generalization for norms that are *locally unitarily invariant* – that is, norms $|||\cdot|||$ on $M_n^{\otimes r}$ that satisfy

$$\big|\big|\big|(U_1 \otimes \cdots \otimes U_r)X(V_1 \otimes \cdots \otimes V_r)\big|\big|\big| = \big|\big|\big|X\big|\big|\big|$$

for all $X \in M_n^{\otimes r}$ and all unitary operators $U_i, V_i \in M_n$ $(1 \leq i \leq r)$. Notice that the norms $\|\cdot\|_{S(k)}$ and $\|\cdot\|_{\gamma,k}$ are all locally unitarily invariant (with $r = 2$), so we believe that norms with this property play an important role in quantum information theory, and hence that the following result is an important first step in the understanding of the general behavior of these norms.

Theorem 3. *Let $|||\cdot|||$ be a norm on $M_n^{\otimes r}$. Then the following are equivalent:*

(a) $|||\cdot|||$ is locally unitarily invariant.

(b) $\big|\big|\big|(A_1 \otimes \cdots \otimes A_r)X(B_1 \otimes \cdots \otimes B_r)\big|\big|\big| \leq \Big(\prod_{i=1}^{r} \|A_i\|\|B_i\|\Big)\big|\big|\big|X\big|\big|\big|$ for all $X \in M_n^{\otimes r}$ and all $A_i, B_i \in M_n$ $(1 \leq i \leq r)$.

Proof. The fact that $(b) \implies (a)$ is straightforward. If we let A_i, B_i $(1 \leq i \leq r)$ be unitary, then

$$\begin{aligned} \big|\big|\big|X\big|\big|\big| &= \big|\big|\big|(A_1^*A_1 \otimes \cdots \otimes A_r^*A_r)X(B_1B_1^* \otimes \cdots \otimes B_rB_r^*)\big|\big|\big| \\ &\leq \big|\big|\big|(A_1 \otimes \cdots \otimes A_r)X(B_1 \otimes \cdots \otimes B_r)\big|\big|\big| \\ &\leq \big|\big|\big|X\big|\big|\big|, \end{aligned}$$

where we used the fact that $\|A_i\| = \|B_i\| = \|A_i^*\| = \|B_i^*\| = 1$ for all i. It follows that $|||(A_1 \otimes \cdots \otimes A_r)X(B_1 \otimes \cdots \otimes B_r)||| = |||X|||$, so $|||\cdot|||$ is locally unitarily invariant.

To see that $(a) \implies (b)$, write $A_i = \sum_j c_{ij}U_{ij}$ and $B_i = \sum_\ell d_{i\ell}V_{i\ell}$ with each U_{ij} and $V_{i\ell}$ unitary (recall that this is possible since the span of the unitary matrices is all of M_n). Then

$$\begin{aligned} &\big|\big|\big|(A_1 \otimes \cdots \otimes A_r)X(B_1 \otimes \cdots \otimes B_r)\big|\big|\big| \\ &= \Big|\Big|\Big|\Big((\sum_j c_{1j}U_{1j}) \otimes \cdots \otimes (\sum_j c_{rj}U_{rj})\Big)X\Big((\sum_\ell d_{1\ell}V_{1\ell}) \otimes \cdots \otimes (\sum_\ell d_{r\ell}V_{r\ell})\Big)\Big|\Big|\Big| \\ &\leq \sum_{\substack{j_1,j_2,\ldots,j_r \\ \ell_1,\ell_2,\ldots,\ell_r}} \Big((\prod_{i=1}^{r} |c_{ij_i}||d_{i\ell_i}|)\big|\big|\big|(U_{1j_1} \otimes \cdots \otimes U_{rj_r})X(V_{1\ell_1} \otimes \cdots \otimes V_{r\ell_r})\big|\big|\big|\Big) \\ &= \sum_{\substack{j_1,j_2,\ldots,j_r \\ \ell_1,\ell_2,\ldots,\ell_r}} (\prod_{i=1}^{r} |c_{ij_i}||d_{i\ell_i}|)\big|\big|\big|X\big|\big|\big| \\ &= \Big(\prod_{i=1}^{r} (\sum_j |c_{ij}|)(\sum_\ell |d_{i\ell}|)\Big)\big|\big|\big|X\big|\big|\big|, \end{aligned}$$

where the inequality follows from the triangle inequality, and the equality between the third and fourth lines follows from local unitary invariance. By taking the infimum over all decompositions of each A_i and B_i of the given form and using Equation (5), the result follows. □

4. The Cross Norm Criterion for Schmidt Number

We begin by showing that the norms (1) and (2) are dual to each other.

Theorem 4. *Let $X \in M_m \otimes M_n$. Then*

$$\big\|X\big\|_{S(k)}^{\circ} = \big\|X\big\|_{\gamma,k}.$$

Proof. Use Theorem 2 with $\mathcal{H} = M_m \otimes M_n$ and $\mathcal{S} = \big\{|v\rangle\langle w| : SR(|v\rangle), SR(|w\rangle) \leq k\big\}$. □

A completely different proof of Theorem 4, based on minimal and maximal operator spaces, was given in [Joh12]. Indeed, the $S(k)$-norm is the k-minimal L^∞-matrix norm on M_n [JKPP11], so the dual of the $S(k)$-norm is analogously the k-maximal L^1-matrix norm on M_n, which can be verified to be $\|\cdot\|_{\gamma,k}$.

As a result of the duality provided by Theorem 4, we are now in a position to provide an elementary proof of a generalization of the fact that a density matrix ρ is separable if and only if $\|\rho\|_{\gamma,1} = 1$ [Rud00], which is known as the *cross norm criterion* for separability. The generalization provided by the following theorem shows that the remaining norms $\|\cdot\|_{\gamma,k}$ characterize Schmidt number in exactly the same way that $\|\cdot\|_{\gamma,1}$ characterizes separability.

Theorem 5 (Generalized Cross Norm Criterion)**.** *Let $\rho \in M_m \otimes M_n$ be a density matrix. Then $SN(\rho) \leq k$ if and only if $\|\rho\|_{\gamma,k} = 1$.*

Proof. Note that $\|\rho\|_{\gamma,k} \geq \|\rho\|_{tr} = 1$ for all ρ, so we only need to show that $SN(\rho) \leq k$ if and only if $\|\rho\|_{\gamma,k} \leq 1$.

The "only if" implication is trivial, since $SN(\rho) \leq k$ implies that we can write $\rho = \sum_i p_i |v_i\rangle\langle v_i|$ with $SR(|v_i\rangle) \leq k$ for all i and $\sum_i p_i = 1$. By comparing this decomposition of ρ with the definition of the norm (2), the implication follows.

For the "if" direction of the proof, suppose $\|\rho\|_{\gamma,k} \leq 1$ and let $Y = Y^* \in M_m \otimes M_n$ be k-block positive. If we write $Y = cI - X$ with X positive semidefinite then $c \geq \|X\|_{S(k)}$, by Proposition 1. We then have

$$\mathrm{Tr}(\rho Y) = \mathrm{Tr}\big(\rho(cI - X)\big) = c - \mathrm{Tr}(\rho X) \geq c - \big\|X\big\|_{S(k)} \geq 0,$$

where we used the duality of Theorem 4 in the first inequality. Since Y is an arbitrary k-block positive operator, it follows that $SN(\rho) \leq k$, which completes the proof. □

5. The Computable Cross Norm Criterion for Schmidt Number

While Theorem 5 is interesting theoretically, it suffers from the problem of being "too strong" of a test for Schmidt number. It determines Schmidt number exactly, so it seems doubtful that we could actually compute the norm $\|\cdot\|_{\gamma,k}$ and put the theorem to use in practice. We thus would like to weaken Theorem 5 in a sense – we would like a similar test for Schmidt number that is easy to apply in practice, at the expense of being weaker (i.e., only being a necessary or sufficient condition for Schmidt number, but not both).

In the $k = 1$ case (i.e., the case of separability), the *computable cross norm criterion* [Rud03] (sometimes called the *realignment criterion* [CW03]) provides exactly such a test. For the remainder of this section, we fix an orthonormal basis $\{|i\rangle\}_{i=1}^n$ of $\mathbb{C}^n$. If we define the *realignment map* $L : M_{m,n} \otimes M_{r,s} \to M_{m,r} \otimes M_{n,s}$ by $L(|i\rangle\langle j| \otimes |k\rangle\langle \ell|) = |i\rangle\langle k| \otimes |j\rangle\langle \ell|$ and extending linearly, it is not difficult to verify that $\|L(X)\|_{\gamma,1} = \|X\|_{\gamma,1}$ for all X. It follows immediately that if ρ is separable then $\|L(\rho)\|_{tr} \leq \|L(\rho)\|_{\gamma,1} = \|\rho\|_{\gamma,1} = 1$, where the final equality comes from Theorem 5.

The fact that separability implies $\|L(\rho)\|_{tr} \leq 1$ is the computable cross norm criterion. In fact, it is straightforward to generalize this result to arbitrary Schmidt number – one can easily show that $\|\rho\|_{\gamma,1} \leq k$ whenever $SN(\rho) \leq k$, which leads to the following test for Schmidt number: if $SN(\rho) \leq k$ then $\|L(\rho)\|_{tr} \leq k$.

However, this generalization of the computable cross norm criterion is not particularly strong. For example, while the computable cross norm criterion is both necessary and sufficient on pure states [CW03, Proposition 1], this generalization is only necessary, but not sufficient, when $k \geq 2$. To see this claim, define

$$|v\rangle := \sqrt{1-(n-1)\varepsilon^2}|1\rangle \otimes |1\rangle + \varepsilon \sum_{i=2}^{n} |i\rangle \otimes |i\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

for some small $\varepsilon > 0$. Clearly $SR(|v\rangle) = n$ regardless of ε, but a simple calculation shows that

$$\big\|L(|v\rangle\langle v|)\big\|_{tr} = \Big((n-1)\varepsilon + \sqrt{1-(n-1)\varepsilon^2}\Big)^2,$$

which decreases toward 1 as $\varepsilon \to 0$. In other words, it is possible that $\|L(|v\rangle\langle v|)\|_{tr} \approx 1$ even though $SR(|v\rangle) = n$, which demonstrates that this generalization of the computable cross norm criterion is not even close to being sufficient, even on pure states.

In order to fix this problem and provide much stronger generalization of the computable cross norm criterion, we again use norm duality techniques. First, consider the following norm on M_n, which can be thought of as a hybrid of the Frobenius norm (i.e., the Schatten 2-norm) and the Ky Fan k-norm:

$$\big\|X\big\|_{(k,2)} := \sup\Big\{\big|\mathrm{Tr}(XY)\big| : \mathrm{rank}(Y) \leq k, \big\|Y\big\|_F \leq 1\Big\} = \sqrt{\sum_{i=1}^{k} \sigma_i^2},$$

where $\|\cdot\|_F$ denotes the Frobenius norm and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the ordered singular values of X. Our generalization of the computable cross norm criterion involves the dual of these norms, which we characterize at the end of this section.

Theorem 6 (Generalized Computable Cross Norm Criterion)**.** *If $\rho \in M_m \otimes M_n$ has $SN(\rho) \leq k$ then $\|L(\rho)\|_{(k^2,2)}^\circ \leq 1$.*

Proof. Suppose $SN(\rho) \leq k$ and begin by writing ρ as a convex combination of projections onto states with Schmidt rank no greater than k:

$$\rho = \sum_i p_i \sum_{j,\ell=1}^{k} \alpha_{ij}\alpha_{i\ell} |v_{ij}\rangle\langle v_{i\ell}| \otimes |w_{ij}\rangle\langle w_{i\ell}|$$

Then

$$L(\rho) = \sum_i p_i \left(\sum_{j=1}^{k} \alpha_{ij} |v_{ij}\rangle\overline{\langle w_{ij}|} \right) \otimes \left(\sum_{\ell=1}^{k} \alpha_{i\ell} \overline{|v_{i\ell}\rangle}\langle w_{i\ell}| \right),$$

where the superscript bars indicate complex conjugation in the same basis used to define the realignment map L.

If we define $A_i := \sum_{j=1}^{k} \alpha_{ij}|v_{ij}\rangle\overline{\langle w_{ij}|}$ then we have $L(\rho) = \sum_i p_i A_i \otimes \overline{A_i}$, where $\mathrm{rank}(A_i) \leq k$ and $\|A_i\|_F = 1$ for all i. In particular then, we have $L(\rho) = \sum_i p_i B_i$, where $\mathrm{rank}(B_i) \leq k^2$ and $\|B_i\|_F = 1$ for all i. Let $|||\cdot|||$ be a norm with the property that $|||X||| = \|X\|_F$ for all X with $\mathrm{rank}(X) \leq k^2$. Then

$$|||L(\rho)||| = \left|\left|\left| \sum_i p_i B_i \right|\right|\right| \leq \sum_i p_i \, |||B_i||| = \sum_i p_i \big\|B_i\big\|_F = \sum_i p_i = 1. \tag{7}$$

All that remains is to make a suitable choice for $|||\cdot|||$, so that this test for Schmidt number is as strong as possible. To this end, notice that $\|\cdot\|_{(k^2,2)}$ is clearly the smallest norm with the required rank property. Also notice that, because the Frobenius norm is self-dual (i.e., $\|\cdot\|_F^\circ = \|\cdot\|_F$), $\|\cdot\|_{(k^2,2)}^\circ$ must satisfy the same rank property, and in particular must be the largest such matrix norm. We thus choose $|||\cdot||| = \|\cdot\|_{(k^2,2)}^\circ$, which completes the proof. □

Notice that when $k = 1$, $\|\cdot\|_{(k^2,2)}$ is the operator norm, so $\|\cdot\|_{(k^2,2)}^\circ = \|\cdot\|_{tr}$ and hence Theorem 6 gives the standard realignment criterion in this case. On the other extreme, if $k = \min\{m,n\}$ then $\|\cdot\|_{(k^2,2)}^\circ = \|\cdot\|_F$. Because L preserves the Frobenius norm, Theorem 6 then simply says that $\|\rho\|_F \leq 1$ for all quantum states ρ, which is trivially true because $\|\rho\|_F \leq \|\rho\|_{tr} = 1$. The conditions given for the remaining values of k are all non-trivial, yet easy to compute.

Also notice that $\|\cdot\|_{(k^2,2)} \leq k\|\cdot\|$, so $\|\cdot\|_{tr} \leq k\|\cdot\|_{(k^2,2)}^\circ$. In other words, Theorem 6 provides a test for Schmidt number that is strictly stronger than the previously-noted criterion that says $\|L(\rho)\|_{tr} \leq k$ whenever $SN(\rho) \leq k$. Furthermore, the test provided by Theorem 6 is strong enough that it is both necessary and sufficient on pure states. That is, $SR(|v\rangle) \leq k$ if and only if $\|L(|v\rangle\langle v|)\|_{(k^2,2)}^\circ \leq 1$. To see this, it suffices to notice that Inequality (7) becomes an equality in this case (since there is only one term in the sum) and $\|X\|_{(k^2,2)}^\circ > 1$ whenever $\|X\|_F = 1$ and $\mathrm{rank}(X) > k^2$.

We now turn our attention to the problem of characterizing the norm $\|\cdot\|_{(k,2)}^\circ$. It follows immediately from Theorem 2 that

$$\big\|X\big\|_{(k,2)}^\circ = \inf\Big\{ \sum_i \big\|Y_i\big\|_F : X = \sum_i Y_i, \text{ with } \mathrm{rank}(Y_i) \leq k \; \forall\, i \Big\},$$

where the infimum is taken over all such decompositions of X. While this characterization is useful theoretically, it provides very little insight into its calculation. Since we claimed that our generalization of the computable cross norm criterion is useful in practice, we need to show how to calculate $\|\cdot\|^{\circ}_{(k^2,2)}$. The following theorem, proved in [MF85], makes this computation explicit.

Theorem 7. *Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be the ordered singular values of $X \in M_n$. Let r be the largest index $1 \leq r < k$ such that $\sigma_r > \sum_{i=r+1}^{\min\{m,n\}} \sigma_i/(k-r)$ (or take $r = 0$ if no such index exists). Also define $\tilde{\sigma} := \sum_{i=r+1}^{\min\{m,n\}} \sigma_i/(k-r)$. Then*

$$\|X\|^{\circ}_{(k,2)} = \sqrt{\sum_{i=1}^{r} \sigma_i^2 + (k-r)\tilde{\sigma}^2}.$$

Using this result, we have tested the criterion provided by Theorem 6 numerically and found that the test is strongest on density matrices of small rank. Indeed, this is expected, as we saw that it is both necessary and sufficient on pure states (i.e., density matrices of rank 1), and the weakness of the criterion comes from Inequality (7), which is weaker when there are more terms in the sum (i.e., when the density matrix has high rank).

More specifically, for each $1 \leq r \leq 36$, we randomly generated 10^6 Haar-uniform pure states in $\mathbb{C}^6 \otimes \mathbb{C}^6 \otimes \mathbb{C}^r$ and traced out the third subsystem, resulting in a random rank-r density matrix in $M_6 \otimes M_6$. The rth row of Table 1 lists the percentage of these states that violated the generalized computable cross norm criterion of Theorem 6 with $k = 1, 2, 3, 4$, and 5.

Observe that every entry in the first row of Table 1 is 100%, a fact that is expected since almost all pure states have maximal Schmidt rank (i.e., Schmidt rank 6) and, as we mentioned earlier, Theorem 6 is both necessary and sufficient in this case. The behavior of the first column is also expected: it was shown in [AN12] that (asymptotically) the computable cross norm criterion typically detects entanglement in states generated in this way when $r \leq (8/3\pi)^2 n^2$, which is approximately equal to 26 in this case.

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	Shown to have Schmidt number greater than...				
Rank	1	2	3	4	5
1	100.00%	100.00%	100.00%	100.00%	100.00%
2	100.00%	100.00%	99.88%	0.00%	0.00%
3	100.00%	100.00%	1.56%	0.00%	0.00%
4	100.00%	100.00%	0.00%	0.00%	0.00%
5	100.00%	100.00%	0.00%	0.00%	0.00%
6	100.00%	99.63%	0.00%	0.00%	0.00%
7	100.00%	21.57%	0.00%	0.00%	0.00%
8	100.00%	0.04%	0.00%	0.00%	0.00%
9–28	100.00%	0.00%	0.00%	0.00%	0.00%
29	99.99%	0.00%	0.00%	0.00%	0.00%
30	99.85%	0.00%	0.00%	0.00%	0.00%
31	98.46%	0.00%	0.00%	0.00%	0.00%
32	91.85%	0.00%	0.00%	0.00%	0.00%
33	74.56%	0.00%	0.00%	0.00%	0.00%
34	48.17%	0.00%	0.00%	0.00%	0.00%
35	23.46%	0.00%	0.00%	0.00%	0.00%
36	8.34%	0.00%	0.00%	0.00%	0.00%

Table 1: A summary of the percentage of 3.6×10^7 randomly-generated density matrices of rank $1 \leq r \leq 36$ in $M_6 \otimes M_6$ that can be shown to have Schmidt number greater than $1 \leq k \leq 5$ by Theorem 6.

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