Duality of Entanglement Norms

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based on joint work with David W. Kribs

WONRA 2012, Kaohsiung, Taiwan

July 11, 2012
Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $F$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- $M_n$, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B \rangle := \text{Tr}(A^\dagger B); \text{ and}$$

- $M_n^H$, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.
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A pure quantum state is a unit vector $|v\rangle \in \mathbb{C}^n$.

A mixed quantum state is a positive semidefinite matrix $\rho \in M_n^H$ with $\text{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

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Quantum States

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$$\rho = \sum_i p_i |v_i\rangle \langle v_i|.$$
A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called **separable** if there exist $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$ so that

$$|v\rangle = |a\rangle \otimes |b\rangle.$$ 

A mixed state $\rho \in M^H_m \otimes M^H_n$ is called **separable** if it can be written as a convex combination of separable pure states:

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with each $|v_i\rangle$ separable.
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Schmidt Decomposition Theorem

**Theorem (Schmidt decomposition)**

For each $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ there exists:

- a positive integer $k \leq \min\{m, n\}$;
- positive real constants $\{\alpha_i\}_{i=1}^k$ with $\sum_{i=1}^k \alpha_i^2 = 1$; and
- orthonormal sets $\{|a_i\rangle\}_{i=1}^k \subset \mathbb{C}^m$ and $\{|b_i\rangle\}_{i=1}^k \subset \mathbb{C}^n$

such that

$$|v\rangle = \sum_{i=1}^k \alpha_i |a_i\rangle \otimes |b_i\rangle.$$
Schmidt Rank

The integer $k$ is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
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Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least $k$ such that $\rho$ can be written as a convex combination of pure states with Schmidt rank $\leq k$:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with} \quad SR(|v_i\rangle) \leq k \quad \text{for all} \quad i.$$

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An operator $X \in M^H_m \otimes M^H_n$ is called $k$-block positive if $\langle v | X | v \rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

- If $X$ is $k$-block positive but not positive semidefinite, it is called a $k$-entanglement witness.
- $SN(\rho) > k$ if and only if there exists a $k$-entanglement witness with $\text{Tr}(X \rho) < 0$.
- The cone of $k$-block positive operators is dual to the set of $\rho$ with $SN(\rho) \leq k$. 
An operator $X \in M_m^H \otimes M_n^H$ is called \textit{k-block positive} if
\[ \langle v | X | v \rangle \geq 0 \text{ for all } |v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n \text{ with } SR(|v\rangle) \leq k. \]

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Block Positivity

An operator $X \in M^H_m \otimes M^H_n$ is called \textbf{$k$-block positive} if $\langle v | X | v \rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

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The **dual** of a norm \( \|\cdot\| \) on \( \mathcal{H} \) is defined as follows:

\[
\|\|v\|\|^\circ := \sup_{w \in \mathcal{H}} \left\{ |\langle w|v \rangle| : \|w\| \leq 1 \right\}.
\]

For example, some important norms on \( M_n \) include...

- the operator norm

\[
\|A\| := \sup \left\{ |\langle v|A|w \rangle| \right\} = \sigma_1(A),
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The **dual** of a norm $\| \cdot \|$ on $\mathcal{H}$ is defined as follows:

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- the **operator norm**
  
  $$\|A\| := \sup \left\{ |\langle v|A|w \rangle| \right\} = \sigma_1(A),$$
the Frobenius norm

$$\|A\|_F := \sqrt{\text{Tr}(A^\dagger A)} = \sqrt{\sum_{i=1}^{n} \sigma_i(A)^2} = \|A\|_F^\circ, \text{ and}$$

the trace norm

$$\|A\|_{tr} := \sum_{i=1}^{n} \sigma_i(A) = \|A\|_F^\circ.$$
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the trace norm

\[ \|A\|_{tr} := \sum_{i=1}^{n} \sigma_i(A) = \|A\|^\circ. \]
Given a fixed $1 \leq k \leq n$, we define the $(k, 2)$-norm on $M_n$ as follows:

$$\|A\|_{(k, 2)} := \sqrt{\sum_{i=1}^{k} \sigma_i(A)^2}.$$ 

- Equals the operator norm when $k = 1$ and the Frobenius norm when $k = n$.
- Their dual norms are a bit of a mouthful...
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Given a fixed $1 \leq k \leq n$, we define the $(k, 2)$-norm on $M_n$ as follows:

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A Ky Fan-Type Duality Result

Theorem

Let $r$ be the largest index $1 \leq r < k$ such that

$$\sigma_r > \sum_{i=r+1}^{\min\{m,n\}} \frac{\sigma_i}{(k - r)} \quad \text{(or take } r = 0 \text{ if no such index exists)}.$$

Also define $\tilde{\sigma} := \sum_{i=r+1}^{\min\{m,n\}} \frac{\sigma_i}{(k - r)}$. Then

$$\|A\|_\circ^{(k,2)} = \sqrt{\sum_{i=1}^{r} \sigma_i^2 + (k - r)\tilde{\sigma}^2}.$$
We now introduce a family of norms that characterize $k$-block positivity. For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\|X\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \left\{ |\langle w | X | v \rangle| : SR(|v\rangle), SR(|w\rangle) \leq k \right\}$$

and

$$r_k(\otimes^\otimes(Y)) := \sup_{|v\rangle} \left\{ |\langle v | Y | v \rangle| : SR(|v\rangle) \leq k \right\}.$$

Any $Z \in M_m^H \otimes M_n^H$ can be written in the form $Z = c1 - X$ for some $X \in (M_m \otimes M_n)^+$. Then $Z$ is $k$-block positive if and only if $c \geq \|X\|_{S(k)} = r_k(\otimes^\otimes(X))$. 
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r_k \otimes (Y) := \sup_{|v\rangle} \left\{ |\langle v | Y | v \rangle| : SR(|v\rangle) \leq k \right\}.
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For $X \in M_m \otimes M_n$ and $Y \in M^H_m \otimes M^H_n$ we define

$$\|X\|_{\gamma,k} := \inf \left\{ \sum_i |c_i| : X = \sum_i c_i |v_i\rangle\langle w_i| \right\},$$

with $SR(|v_i\rangle), SR(|w_i\rangle) \leq k \ \forall \ i$, and

$$R_k(Y) := \inf \left\{ \sum_i |c_i| : Y = \sum_i c_i |v_i\rangle\langle v_i| \text{ with } SR(|v_i\rangle) \leq k \ \forall \ i \right\}.$$
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In the $k = 1$ case, we have

$$\|X\|_{\gamma,1} := \inf \left\{ \sum_i \|A_i\|_{tr} \|B_i\|_{tr} : X = \sum_i A_i \otimes B_i \right\}.$$ 

- Rudolph showed (2000) that $\rho$ is separable if and only if $\|\rho\|_{\gamma,1} = 1$ (this is the cross norm criterion for separability).

- Also, $R_1(\rho) = 2E_R(\rho) + 1$, where $E_R$ is the robustness of entanglement:

$$E_R(\rho) := \inf \{ s : \rho + s\sigma \text{ is separable} \},$$

where the infimum is taken over all separable $\sigma$. 

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Theorem

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$$\|X\|_{S(k)}^\circ = \|X\|_{\gamma,k} \quad \text{and} \quad r_k^\otimes (Y)^\circ = R_k(Y).$$
Generalizing the Cross Norm Criterion

Theorem

Let $\rho \in M_m \otimes M_n$ be a density matrix. Then $SN(\rho) \leq k$ if and only if $\|\rho\|_{\gamma,k} = 1$ if and only if $R_k(\rho) = 1$.

The proof is elementary and only a couple lines long.
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Values on Pure States

Schmidt number is easy to determine for pure states, so we might hope that these norms are easy to compute for pure states too.

Suppose $|v\rangle$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Then

$$\| |v\rangle\langle v| \|_{S(k)} = \sum_{i=1}^{k} \alpha_i^2.$$
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Values on Pure States

From the Ky Fan-type duality result from earlier, we get 
\[ \|\|v\rangle\langle v\|\|_{\gamma,k} \]. In particular, let \( r \) be the largest index \( 1 \leq r < k \) such that 
\[ \alpha_r > \sum_{i=r+1}^{\min\{m,n\}} \alpha_i / (k - r) \] (or take \( r = 0 \) if no such index exists) and define 
\[ \tilde{\alpha} := \sum_{i=r+1}^{\min\{m,n\}} \alpha_i / (k - r) \]. Then

\[ \|\|v\rangle\langle v\|\|_{\gamma,k} = \sum_{i=1}^{r} \alpha_i^2 + (k - r)\tilde{\alpha}^2. \]

When \( k = 1 \), this simplifies to

\[ \|\|v\rangle\langle v\|\|_{\gamma,1} = \left( \sum_{i=1}^{\min\{m,n\}} \alpha_i \right)^2. \]
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An Open Question

What about $R_k(|v\rangle\langle v|)$? We don’t know!

Our best guess is that $R_k(|v\rangle\langle v|) = 2\|v\rangle\langle v\|_{\gamma,k} - 1$. 

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The realignment map is the linear map $L : M_n \otimes M_n \to M_n \otimes M_n$ defined by $L(|i\rangle\langle j| \otimes |k\rangle\langle l|) = |i\rangle\langle k| \otimes |j\rangle\langle l|.$

The realignment criterion for separability says that if $\rho$ is separable, then $\|L(\rho)\|_{tr} \leq 1$. How does this generalize?

The “easy” generalization is that if $SN(\rho) \leq k$ then $\|L(\rho)\|_{tr} \leq k$. This is true! But it is unsatisfying...
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Theorem

If $\rho \in M_m \otimes M_n$ has $SN(\rho) \leq k$ then $\|L(\rho)\|_{(k^2,2)}^\circ \leq 1$.

- Strictly stronger than the $\|L(\rho)\|_{tr} \leq k$ criterion.
- It is both necessary and sufficient for pure states.
Generalizing the Realignment Criterion

**Theorem**

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