Duality of Entanglement Norms

Nathaniel Johnston
based on joint work with David W. Kribs

WONRA 2012, Kaohsiung, Taiwan

July 11, 2012

Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- M_n, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B\rangle := \mathrm{Tr}(A^\dagger B); \text{ and}$$

- M_n^H, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.

Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- M_n, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B \rangle := \mathrm{Tr}(A^\dagger B); \text{ and}$$

- M_n^H, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.

Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- M_n, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B \rangle := \mathrm{Tr}(A^\dagger B); \text{ and}$$

- M_n^H, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.

Our Notation and Setting

We use $\mathcal{H}$ to denote a finite-dimensional Hilbert space over a field $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$). Some examples...

- $\mathbb{C}^n$, complex Euclidean space with the usual inner product;
- M_n, the $n \times n$ complex matrices with the Hilbert–Schmidt inner product

$$\langle A|B\rangle := \mathrm{Tr}(A^\dagger B); \text{ and}$$

- M_n^H, the $n \times n$ complex Hermitian matrices, also with the Hilbert–Schmidt inner product.

Quantum States

A **pure quantum state** is a unit vector $|v\rangle \in \mathbb{C}^n$.

A **mixed quantum state** is a positive semidefinite matrix $\rho \in M_n^H$ with $\mathrm{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|.$$

Quantum States

A **pure quantum state** is a unit vector $|v\rangle \in \mathbb{C}^n$.

A **mixed quantum state** is a positive semidefinite matrix $\rho \in M_n^H$ with $\mathrm{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|.$$

Quantum States

A **pure quantum state** is a unit vector $|v\rangle \in \mathbb{C}^n$.

A **mixed quantum state** is a positive semidefinite matrix $\rho \in M_n^H$ with $\mathrm{Tr}(\rho) = 1$.

Mixed states can be written as convex combinations of projections onto pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i|.$$

Separability and Entanglement

A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called separable if there exist $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$ so that

$$|v\rangle = |a\rangle \otimes |b\rangle.$$

A mixed state $\rho \in M_m^H \otimes M_n^H$ is called separable if it can be written as a convex combination of separable pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with each } |v_i\rangle \text{ separable.}$$

Separability and Entanglement

A pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called separable if there exist $|a\rangle \in \mathbb{C}^m$ and $|b\rangle \in \mathbb{C}^n$ so that

$$|v\rangle = |a\rangle \otimes |b\rangle.$$

A mixed state $\rho \in M_m^H \otimes M_n^H$ is called separable if it can be written as a convex combination of separable pure states:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with each } |v_i\rangle \text{ separable.}$$

Schmidt Decomposition Theorem

Theorem (Schmidt decomposition)

For each $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ *there exists:*

- *a positive integer* $k \leq \min\{m, n\}$;
- *positive real constants* $\{\alpha_i\}_{i=1}^k$ *with* $\sum_{i=1}^k \alpha_i^2 = 1$; *and*
- *orthonormal sets* $\{|a_i\rangle\}_{i=1}^k \subset \mathbb{C}^m$ *and* $\{|b_i\rangle\}_{i=1}^k \subset \mathbb{C}^n$

such that

$$|v\rangle = \sum_{i=1}^k \alpha_i |a_i\rangle \otimes |b_i\rangle.$$

Schmidt Rank

The integer k is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^k$ are called the **Schmidt coefficients** of $|v\rangle$.

Schmidt Rank

The integer k is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^k$ are called the **Schmidt coefficients** of $|v\rangle$.

Schmidt Rank

The integer k is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^{k}$ are called the **Schmidt coefficients** of $|v\rangle$.

Schmidt Rank

The integer k is called the **Schmidt rank** of $|v\rangle$, denoted $SR(|v\rangle)$.

- $SR(|v\rangle) = 1$ if and only if $|v\rangle$ is separable.
- If $SR(|v\rangle) \geq 2$ then $|v\rangle$ is called **entangled**.
- The constants $\{\alpha_i\}_{i=1}^k$ are called the **Schmidt coefficients** of $|v\rangle$.

Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least k such that ρ can be written as a convex combination of pure states with Schmidt rank $\leq k$:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \text{ with } SR(|v_i\rangle) \leq k \text{ for all } i.$$

- $SN(\rho) = 1$ if and only if ρ is separable.
- If $SN(\rho) \geq 2$ then ρ is called **entangled**.
- $SN(|v\rangle\langle v|) = SR(|v\rangle)$ for all $|v\rangle$.

Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least k such that ρ can be written as a convex combination of pure states with Schmidt rank $\leq k$:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with } SR(|v_i\rangle) \leq k \text{ for all } i.$$

- $SN(\rho) = 1$ if and only if ρ is separable.
- If $SN(\rho) \geq 2$ then ρ is called **entangled**.
- $SN(|v\rangle\langle v|) = SR(|v\rangle)$ for all $|v\rangle$.

Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least k such that ρ can be written as a convex combination of pure states with Schmidt rank $\leq k$:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with } SR(|v_i\rangle) \leq k \text{ for all } i.$$

- $SN(\rho) = 1$ if and only if ρ is separable.
- If $SN(\rho) \geq 2$ then ρ is called **entangled**.
- $SN(|v\rangle\langle v|) = SR(|v\rangle)$ for all $|v\rangle$.

Schmidt Number

The **Schmidt number** of a mixed state $\rho \in M_m^H \otimes M_n^H$, denoted $SN(\rho)$, is the least k such that ρ can be written as a convex combination of pure states with Schmidt rank $\leq k$:

$$\rho = \sum_i p_i |v_i\rangle\langle v_i| \quad \text{with } SR(|v_i\rangle) \leq k \text{ for all } i.$$

- $SN(\rho) = 1$ if and only if ρ is separable.
- If $SN(\rho) \geq 2$ then ρ is called **entangled**.
- $SN(|v\rangle\langle v|) = SR(|v\rangle)$ for all $|v\rangle$.

Block Positivity

An operator $X \in M_m^H \otimes M_n^H$ is called **k-block positive** if $\langle v|X|v\rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

- If X is k-block positive but not positive semidefinite, it is called a **k-entanglement witness**.
- $SN(\rho) > k$ if and only if there exists a k-entanglement witness with $\mathrm{Tr}(X\rho) < 0$.
- The cone of k-block positive operators is dual to the set of ρ with $SN(\rho) \leq k$.

Block Positivity

An operator $X \in M_m^H \otimes M_n^H$ is called **k-block positive** if $\langle v|X|v\rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

- If X is k-block positive but not positive semidefinite, it is called a **k-entanglement witness**.
- $SN(\rho) > k$ if and only if there exists a k-entanglement witness with $\mathrm{Tr}(X\rho) < 0$.
- The cone of k-block positive operators is dual to the set of ρ with $SN(\rho) \leq k$.

Block Positivity

An operator $X \in M_m^H \otimes M_n^H$ is called **k-block positive** if $\langle v|X|v\rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

- If X is k-block positive but not positive semidefinite, it is called a **k-entanglement witness**.
- $SN(\rho) > k$ if and only if there exists a k-entanglement witness with $\mathrm{Tr}(X\rho) < 0$.
- The cone of k-block positive operators is dual to the set of ρ with $SN(\rho) \leq k$.

Block Positivity

An operator $X \in M_m^H \otimes M_n^H$ is called **k-block positive** if $\langle v|X|v\rangle \geq 0$ for all $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $SR(|v\rangle) \leq k$.

- If X is k-block positive but not positive semidefinite, it is called a **k-entanglement witness**.
- $SN(\rho) > k$ if and only if there exists a k-entanglement witness with $\mathrm{Tr}(X\rho) < 0$.
- The cone of k-block positive operators is dual to the set of ρ with $SN(\rho) \leq k$.

Dual Norms

The **dual** of a norm $|||\cdot|||$ on $\mathcal{H}$ is defined as follows:

$$|||\mathbf{v}|||^\circ := \sup_{\mathbf{w}\in\mathcal{H}} \Big\{ \big|\langle \mathbf{w}|\mathbf{v}\rangle\big| : |||\mathbf{w}||| \leq 1 \Big\}.$$

For example, some important norms on M_n include...

- the operator norm

$$\big\|A\big\| := \sup \Big\{ \big|\langle v|A|w\rangle\big| \Big\} = \sigma_1(A),$$

Dual Norms

The **dual** of a norm $|||\cdot|||$ on $\mathcal{H}$ is defined as follows:

$$|||\mathbf{v}|||^\circ := \sup_{\mathbf{w}\in\mathcal{H}} \Big\{ \big| \langle \mathbf{w}|\mathbf{v}\rangle \big| : |||\mathbf{w}||| \leq 1 \Big\}.$$

For example, some important norms on M_n include...

- the **operator norm**

$$\big\|A\big\| := \sup \Big\{ \big| \langle v|A|w\rangle \big| \Big\} = \sigma_1(A),$$

Dual Norms

- the **Frobenius norm**

$$\|A\|_F := \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum_{i=1}^{n} \sigma_i(A)^2} = \|A\|_F^\circ, \text{ and}$$

- the **trace norm**

$$\|A\|_{tr} := \sum_{i=1}^{n} \sigma_i(A) = \|A\|^\circ.$$

Dual Norms

- the **Frobenius norm**

$$\big\|A\big\|_F := \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum_{i=1}^{n}\sigma_i(A)^2} = \big\|A\big\|_F^\circ, \text{ and}$$

- the **trace norm**

$$\big\|A\big\|_{tr} := \sum_{i=1}^{n}\sigma_i(A) = \big\|A\big\|^\circ.$$

A Ky Fan-Type Duality Result

Given a fixed $1 \leq k \leq n$, we define the $(k,2)$-norm on M_n as follows:

$$\big\|A\big\|_{(k,2)} := \sqrt{\sum_{i=1}^{k} \sigma_i(A)^2}.$$

- Equals the operator norm when $k = 1$ and the Frobenius norm when $k = n$.
- Their dual norms are a bit of a mouthful...

A Ky Fan-Type Duality Result

Given a fixed $1 \leq k \leq n$, we define the $(k,2)$-norm on M_n as follows:

$$\big\|A\big\|_{(k,2)} := \sqrt{\sum_{i=1}^{k} \sigma_i(A)^2}.$$

- Equals the operator norm when $k = 1$ and the Frobenius norm when $k = n$.
- Their dual norms are a bit of a mouthful...

A Ky Fan-Type Duality Result

Given a fixed $1 \leq k \leq n$, we define the $(k, 2)$-norm on M_n as follows:

$$\big\|A\big\|_{(k,2)} := \sqrt{\sum_{i=1}^{k} \sigma_i(A)^2}.$$

- Equals the operator norm when $k = 1$ and the Frobenius norm when $k = n$.
- Their dual norms are a bit of a mouthful...

A Ky Fan-Type Duality Result

Theorem

Let r be the largest index $1 \leq r < k$ such that $\sigma_r > \sum_{i=r+1}^{\min\{m,n\}} \sigma_i/(k-r)$ (or take $r = 0$ if no such index exists). Also define $\tilde{\sigma} := \sum_{i=r+1}^{\min\{m,n\}} \sigma_i/(k-r)$. Then

$$\big\|A\big\|_{(k,2)}^{\circ} = \sqrt{\sum_{i=1}^{r} \sigma_i^2 + (k-r)\tilde{\sigma}^2}.$$

$S(k)$-Norms and Product Numerical Radius

We now introduce a family of norms that characterize k-block positivity. For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\} \text{ and}$$

$$r_k^{\otimes}(Y) := \sup_{|v\rangle} \Big\{ \big|\langle v|Y|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

- Any $Z \in M_m^H \otimes M_n^H$ can be written in the form $Z = cI - X$ for some $X \in (M_m \otimes M_n)^+$. Then Z is k-block positive if and only if $c \geq \|X\|_{S(k)} = r_k^{\otimes}(X)$.

$S(k)$-Norms and Product Numerical Radius

We now introduce a family of norms that characterize k-block positivity. For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\} \text{ and}$$

$$r_k^{\otimes}(Y) := \sup_{|v\rangle} \Big\{ \big|\langle v|Y|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

- Any $Z \in M_m^H \otimes M_n^H$ can be written in the form $Z = cI - X$ for some $X \in (M_m \otimes M_n)^+$. Then Z is k-block positive if and only if $c \geq \|X\|_{S(k)} = r_k^{\otimes}(X)$.

$S(k)$-Norms and Product Numerical Radius

We now introduce a family of norms that characterize k-block positivity. For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\big\|X\big\|_{S(k)} := \sup_{|v\rangle,|w\rangle} \Big\{ \big|\langle w|X|v\rangle\big| : SR(|v\rangle), SR(|w\rangle) \leq k \Big\} \text{ and}$$

$$r_k^{\otimes}(Y) := \sup_{|v\rangle} \Big\{ \big|\langle v|Y|v\rangle\big| : SR(|v\rangle) \leq k \Big\}.$$

- Any $Z \in M_m^H \otimes M_n^H$ can be written in the form $Z = cI - X$ for some $X \in (M_m \otimes M_n)^+$. Then Z is k-block positive if and only if $c \geq \|X\|_{S(k)} = r_k^{\otimes}(X)$.

Projective Tensor Norm and Robustness of Entanglement

For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\big\|X\big\|_{\gamma,k} := \inf\Big\{ \sum_i |c_i| : X = \sum_i c_i |v_i\rangle\langle w_i| \text{ with } SR(|v_i\rangle), SR(|w_i\rangle) \leq k \ \forall\, i \Big\}, \text{ and}$$

$$R_k(Y) := \inf\Big\{ \sum_i |c_i| : Y = \sum_i c_i |v_i\rangle\langle v_i| \text{ with } SR(|v_i\rangle) \leq k \ \forall\, i \Big\}.$$

Projective Tensor Norm and Robustness of Entanglement

For $X \in M_m \otimes M_n$ and $Y \in M_m^H \otimes M_n^H$ we define

$$\big\|X\big\|_{\gamma,k} := \inf\Big\{ \sum_i |c_i| : X = \sum_i c_i |v_i\rangle\langle w_i| \text{ with } SR(|v_i\rangle), SR(|w_i\rangle) \leq k \;\forall\, i \Big\}, \text{ and}$$

$$R_k(Y) := \inf\Big\{ \sum_i |c_i| : Y = \sum_i c_i |v_i\rangle\langle v_i| \text{ with } SR(|v_i\rangle) \leq k \;\forall\, i \Big\}.$$

Projective Tensor Norm and Robustness of Entanglement

In the $k = 1$ case, we have

$$\big\|X\big\|_{\gamma,1} := \inf\left\{\sum_i \big\|A_i\big\|_{tr}\big\|B_i\big\|_{tr} : X = \sum_i A_i \otimes B_i\right\}.$$

- Rudolph showed (2000) that ρ is separable if and only if $\|\rho\|_{\gamma,1} = 1$ (this is the cross norm criterion for separability).
- Also, $R_1(\rho) = 2E_R(\rho) + 1$, where E_R is the robustness of entanglement:

$$E_R(\rho) := \inf\big\{s : \rho + s\sigma \text{ is separable}\big\},$$

where the infimum is taken over all separable σ.

Projective Tensor Norm and Robustness of Entanglement

In the $k = 1$ case, we have

$$\big\|X\big\|_{\gamma,1} := \inf\Big\{\sum_i \big\|A_i\big\|_{tr}\big\|B_i\big\|_{tr} : X = \sum_i A_i \otimes B_i\Big\}.$$

- Rudolph showed (2000) that ρ is separable if and only if $\|\rho\|_{\gamma,1} = 1$ (this is the cross norm criterion for separability).
- Also, $R_1(\rho) = 2E_R(\rho) + 1$, where E_R is the robustness of entanglement:

$$E_R(\rho) := \inf\big\{s : \rho + s\sigma \text{ is separable}\big\},$$

where the infimum is taken over all separable σ.

Projective Tensor Norm and Robustness of Entanglement

In the $k = 1$ case, we have

$$\|X\|_{\gamma,1} := \inf\Big\{ \sum_i \|A_i\|_{tr} \|B_i\|_{tr} : X = \sum_i A_i \otimes B_i \Big\}.$$

- Rudolph showed (2000) that ρ is separable if and only if $\|\rho\|_{\gamma,1} = 1$ (this is the **cross norm criterion** for separability).
- Also, $R_1(\rho) = 2E_R(\rho) + 1$, where E_R is the **robustness of entanglement**:

$$E_R(\rho) := \inf\big\{ s : \rho + s\sigma \text{ is separable} \big\},$$

where the infimum is taken over all separable σ.

Duality of Entanglement Norms

Theorem

Let $X \in M_m \otimes M_n$ *and* $Y \in M_m^H \otimes M_n^H$*. Then*

$$\big\|X\big\|_{S(k)}^{\circ} = \big\|X\big\|_{\gamma,k} \quad \text{and} \quad r_k^{\otimes}(Y)^{\circ} = R_k(Y).$$

Generalizing the Cross Norm Criterion

Theorem

Let $\rho \in M_m \otimes M_n$ *be a density matrix. Then* $SN(\rho) \leq k$ *if and only if* $\|\rho\|_{\gamma,k} = 1$ *if and only if* $R_k(\rho) = 1$.

- The proof is elementary and only a couple lines long.

Generalizing the Cross Norm Criterion

Theorem

Let $\rho \in M_m \otimes M_n$ be a density matrix. Then $SN(\rho) \leq k$ if and only if $\|\rho\|_{\gamma,k} = 1$ if and only if $R_k(\rho) = 1$.

- The proof is elementary and only a couple lines long.

Values on Pure States

Schmidt number is easy to determine for pure states, so we might hope that these norms are easy to compute for pure states too.

Suppose $|v\rangle$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Then

$$\big\| |v\rangle\langle v| \big\|_{S(k)} = \sum_{i=1}^{k} \alpha_i^2.$$

Values on Pure States

Schmidt number is easy to determine for pure states, so we might hope that these norms are easy to compute for pure states too.

Suppose $|v\rangle$ has Schmidt coefficients $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$. Then

$$\big\| |v\rangle\langle v| \big\|_{S(k)} = \sum_{i=1}^{k} \alpha_i^2.$$

Values on Pure States

From the Ky Fan-type duality result from earlier, we get $\big\||v\rangle\langle v|\big\|_{\gamma,k}$. In particular, let r be the largest index $1 \leq r < k$ such that $\alpha_r > \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$ (or take $r = 0$ if no such index exists) and define $\tilde{\alpha} := \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$. Then

$$\big\||v\rangle\langle v|\big\|_{\gamma,k} = \sum_{i=1}^{r} \alpha_i^2 + (k-r)\tilde{\alpha}^2.$$

When $k = 1$, this simiplifies to

$$\big\||v\rangle\langle v|\big\|_{\gamma,1} = \left(\sum_{i=1}^{\min\{m,n\}} \alpha_i\right)^2.$$

Values on Pure States

From the Ky Fan-type duality result from earlier, we get $\big\| |v\rangle\langle v| \big\|_{\gamma,k}$. In particular, let r be the largest index $1 \leq r < k$ such that $\alpha_r > \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$ (or take $r = 0$ if no such index exists) and define $\tilde{\alpha} := \sum_{i=r+1}^{\min\{m,n\}} \alpha_i/(k-r)$. Then

$$\big\| |v\rangle\langle v| \big\|_{\gamma,k} = \sum_{i=1}^{r} \alpha_i^2 + (k-r)\tilde{\alpha}^2.$$

When $k = 1$, this simiplifies to

$$\big\| |v\rangle\langle v| \big\|_{\gamma,1} = \left(\sum_{i=1}^{\min\{m,n\}} \alpha_i \right)^2.$$

An Open Question

What about $R_k(|v\rangle\langle v|)$? We don't know!

Our best guess is that $R_k(|v\rangle\langle v|) = 2\big\| |v\rangle\langle v| \big\|_{\gamma,k} - 1$.

An Open Question

What about $R_k(|v\rangle\langle v|)$? We don't know!

Our best guess is that $R_k(|v\rangle\langle v|) = 2\big\||v\rangle\langle v|\big\|_{\gamma,k} - 1$.

Generalizing the Realignment Criterion

The **realignment map** is the linear map $L : M_n \otimes M_n \rightarrow M_n \otimes M_n$ defined by $L(|i\rangle\langle j| \otimes |k\rangle\langle \ell|) = |i\rangle\langle k| \otimes |j\rangle\langle \ell|$.

The **realignment criterion** for separability says that if ρ is separable, then $\|L(\rho)\|_{tr} \leq 1$. How does this generalize?

The "easy" generalization is that if $SN(\rho) \leq k$ then $\|L(\rho)\|_{tr} \leq k$. This is true! But it is unsatisfying...

Generalizing the Realignment Criterion

The **realignment map** is the linear map $L : M_n \otimes M_n \to M_n \otimes M_n$ defined by $L(|i\rangle\langle j| \otimes |k\rangle\langle \ell|) = |i\rangle\langle k| \otimes |j\rangle\langle \ell|$.

The **realignment criterion** for separability says that if ρ is separable, then $\|L(\rho)\|_{tr} \leq 1$. How does this generalize?

The "easy" generalization is that if $SN(\rho) \leq k$ then $\|L(\rho)\|_{tr} \leq k$. This is true! But it is unsatisfying...

Generalizing the Realignment Criterion

The **realignment map** is the linear map $L : M_n \otimes M_n \rightarrow M_n \otimes M_n$ defined by $L(|i\rangle\langle j| \otimes |k\rangle\langle \ell|) = |i\rangle\langle k| \otimes |j\rangle\langle \ell|$.

The **realignment criterion** for separability says that if ρ is separable, then $\|L(\rho)\|_{tr} \leq 1$. How does this generalize?

The "easy" generalization is that if $SN(\rho) \leq k$ then $\|L(\rho)\|_{tr} \leq k$. This is true! But it is unsatisfying...

Generalizing the Realignment Criterion

Theorem

If $\rho \in M_m \otimes M_n$ *has* $SN(\rho) \leq k$ *then* $\|L(\rho)\|^{\circ}_{(k^2,2)} \leq 1$.

- Strictly stronger than the $\|L(\rho)\|_{tr} \leq k$ criterion.
- It is both necessary and sufficient for pure states.

Generalizing the Realignment Criterion

Theorem

If $\rho \in M_m \otimes M_n$ has $SN(\rho) \leq k$ then $\|L(\rho)\|^{\circ}_{(k^2,2)} \leq 1$.

- Strictly stronger than the $\|L(\rho)\|_{tr} \leq k$ criterion.
- It is both necessary and sufficient for pure states.

Generalizing the Realignment Criterion

Theorem

If $\rho \in M_m \otimes M_n$ has $SN(\rho) \leq k$ then $\|L(\rho)\|^{\circ}_{(k^2,2)} \leq 1$.

- Strictly stronger than the $\|L(\rho)\|_{tr} \leq k$ criterion.
- It is both necessary and sufficient for pure states.

Bibliography

- Z. Puchała, P. Gawron, J. A. Miszczak, Ł. Skowronek, M.-D. Choi, and K. Życzkowski. Product numerical range in a space with tensor product structure. *Linear Algebra Appl.*, 434:327–342, 2011.
- N. Johnston and D. W. Kribs. A family of norms with applications in quantum information theory. *J. Math. Phys.*, 51:082202, 2010.
- O. Rudolph. A separability criterion for density operators. *J. Phys. A: Math. Gen.*, 33:3951–3955, 2000.
- G. Vidal and R. Tarrach. Robustness of entanglement. *Phys. Rev. A*, 59:141–155, 1999.