Process Tomography for Unitary Quantum Channels

Gus Gutoski and **Nathaniel Johnston**

QuIGS Seminar, Guelph, Ontario

September 24, 2013

Introduction to State Tomography

Suppose we have a quantum state ρ (say as the output of some experimental procedure) and we would like to obtain a mathematical description of that state.

Solution: measure that state repeatedly. Measuring ρ according to the observable A gives us information about $\mathrm{Tr}(A\rho)$.

Each observable can only give us information about a 1-dimensional projection of ρ, so we use many different observables.

Introduction to State Tomography

Suppose we have a quantum state ρ (say as the output of some experimental procedure) and we would like to obtain a mathematical description of that state.

Solution: measure that state repeatedly. Measuring ρ according to the observable A gives us information about $\mathrm{Tr}(A\rho)$.

Each observable can only give us information about a 1-dimensional projection of ρ, so we use many different observables.

Introduction to State Tomography

Suppose we have a quantum state ρ (say as the output of some experimental procedure) and we would like to obtain a mathematical description of that state.

Solution: measure that state repeatedly. Measuring ρ according to the observable A gives us information about $\mathrm{Tr}(A\rho)$.

Each observable can only give us information about a 1-dimensional projection of ρ, so we use many different observables.

Introduction to State Tomography

We have some tuple of *m* observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the state ρ, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(\rho) := (\mathrm{Tr}(A_1\rho), \mathrm{Tr}(A_2\rho), \ldots, \mathrm{Tr}(A_m\rho)).$$

Introduction to State Tomography

We have some tuple of m observables:

$$\mathbf{A} := (A_1, A_2, \ldots, A_m).$$

If our quantum system is in the state ρ, then measuring $\mathbf{A}$ gives the average values

$$\mathbf{A}(\rho) := (\mathrm{Tr}(A_1\rho), \mathrm{Tr}(A_2\rho), \ldots, \mathrm{Tr}(A_m\rho)).$$

Tomography of Pure States

We want ρ to be uniquely determined by $\mathbf{A}(\rho)$. If $\rho \in M_B$, where B is a d-dimensional quantum system, we need $d^2 - 1$ observables to completely reconstruct ρ.

Question

What if ρ is a pure state?

In this case, we can do much better: we can reconstruct pure states with only $O(d)$ observables. More specifically...

Tomography of Pure States

We want ρ to be uniquely determined by $\mathbf{A}(\rho)$. If $\rho \in M_B$, where B is a d-dimensional quantum system, we need $d^2 - 1$ observables to completely reconstruct ρ.

Question

What if ρ is a pure state?

In this case, we can do much better: we can reconstruct pure states with only $O(d)$ observables. More specifically...

Tomography of Pure States

Theorem (Heinosaari–Mazzarella–Wolf)

There exists a set of $m = 4d - 5$ observables for which no two pure states have the same measurement results.

Theorem (Chen–Dawkins–Ji–J.–Kribs–Shultz–Zeng)

There exists a set of $m = 5d - 7$ observables for which every pure state is such that no other state (pure or mixed) has the same measurement results.

[1] T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. *Comm. Math. Phys.*, 318:355–374, 2013.

[2] J. Chen, H. Dawkins, Z. Ji, N. J., D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. *Phys. Rev. A*, 88:012109, 2013.

Tomography of Pure States

Theorem (Heinosaari–Mazzarella–Wolf)

There exists a set of $m = 4d - 5$ observables for which no two pure states have the same measurement results.

Theorem (Chen–Dawkins–Ji–J.–Kribs–Shultz–Zeng)

There exists a set of $m = 5d - 7$ observables for which every pure state is such that no other state (pure or mixed) has the same measurement results.

[1] T. Heinosaari, L. Mazzarella, and M. M. Wolf. Quantum tomography under prior information. *Comm. Math. Phys.*, 318:355–374, 2013.

[2] J. Chen, H. Dawkins, Z. Ji, N. J., D. W. Kribs, F. Shultz, and B. Zeng. Uniqueness of quantum states compatible with given measurement results. *Phys. Rev. A*, 88:012109, 2013.

How It's Done: HMW Theorem

Theorem (Heinosaari–Mazzarella–Wolf)

There exists a set of $m = 4d - 5$ observables for which no two pure states have the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: HMW Theorem

Theorem (Heinosaari–Mazzarella–Wolf)

There exists a set of $m = 4d - 5$ observables for which no two pure states have the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| \in \mathbf{A}^{\perp}$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: HMW Theorem

Theorem (Heinosaari–Mazzarella–Wolf)

There exists a set of $m = 4d - 5$ observables for which no two pure states have the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and $|\psi\rangle\langle\psi|$ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi| \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: HMW Theorem

Notice that $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$ always has rank 2. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with rank ≥ 3.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$?

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (4d - 5) = (d - 2)^2$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: HMW Theorem

Notice that $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$ always has rank 2. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with rank ≥ 3.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$?

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (4d - 5) = (d - 2)^2$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: HMW Theorem

Notice that $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$ always has rank 2. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with rank ≥ 3.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$?

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (4d - 5) = (d - 2)^2$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: HMW Theorem

Notice that $|\phi\rangle\langle\phi| - |\psi\rangle\langle\psi|$ always has rank 2. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with rank ≥ 3.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ *of the Hermitian traceless matrices so that every* $0 \neq X \in \mathbf{A}^\perp$ *has* $\mathrm{rank}(X) \geq 3$*?*

The family of $4d - 5$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (4d - 5) = (d - 2)^2$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: HMW Theorem

The basic idea is to construct a basis of $\mathbf{A}^\perp$ along diagonals. An example in the $d = 4$ case is:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & -2 \end{bmatrix}$$

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & i & \cdot & \cdot \\ -i & \cdot & i & \cdot \\ \cdot & -i & \cdot & i \\ \cdot & \cdot & -i & \cdot \end{bmatrix}$$

How It's Done: HMW Theorem

The basic idea is to construct a basis of $\mathbf{A}^{\perp}$ along diagonals. An example in the $d = 4$ case is:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & -2 \end{bmatrix}$$

$$\begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & i & \cdot & \cdot \\ -i & \cdot & i & \cdot \\ \cdot & -i & \cdot & i \\ \cdot & \cdot & -i & \cdot \end{bmatrix}$$

How It's Done: CDJJKSZ Theorem

Theorem (Chen–Dawkins–Ji–J.–Kribs–Shultz–Zeng)

There exists a set of $m = 5d - 7$ observables for which every pure state is such that no other state (pure or mixed) has the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and σ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(\sigma - |\phi\rangle\langle\phi|)\big) = 0 \quad \forall 1 \leq j \leq m.$$

That is, $\sigma - |\phi\rangle\langle\phi| \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: CDJJKSZ Theorem

Theorem (Chen–Dawkins–Ji–J.–Kribs–Shultz–Zeng)

There exists a set of $m = 5d - 7$ observables for which every pure state is such that no other state (pure or mixed) has the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and σ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(\sigma - |\phi\rangle\langle\phi|)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $\sigma - |\phi\rangle\langle\phi| \in \mathbf{A}^{\perp}$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: CDJJKSZ Theorem

Theorem (Chen–Dawkins–Ji–J.–Kribs–Shultz–Zeng)

There exists a set of $m = 5d - 7$ observables for which every pure state is such that no other state (pure or mixed) has the same measurement results.

Key idea: Suppose $|\phi\rangle\langle\phi|$ and σ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(\sigma - |\phi\rangle\langle\phi|)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $\sigma - |\phi\rangle\langle\phi| \in \mathbf{A}^{\perp}$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done: CDJJKSZ Theorem

Notice that $\sigma - |\phi\rangle\langle\phi|$ always has at most 1 negative eigenvalue. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with 2 or more negative eigenvalues.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues?

The family of $5d - 7$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (5d - 7) = (d - 2)(d - 3)$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: CDJJKSZ Theorem

Notice that $\sigma - |\phi\rangle\langle\phi|$ always has at most 1 negative eigenvalue. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with 2 or more negative eigenvalues.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues?

The family of $5d - 7$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (5d - 7) = (d - 2)(d - 3)$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: CDJJKSZ Theorem

Notice that $\sigma - |\phi\rangle\langle\phi|$ always has at most 1 negative eigenvalue. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with 2 or more negative eigenvalues.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues?

The family of $5d-7$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2-1)-(5d-7)=(d-2)(d-3)$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: CDJJKSZ Theorem

Notice that $\sigma - |\phi\rangle\langle\phi|$ always has at most 1 negative eigenvalue. So we can just look for a large subspace of matrices $\mathbf{A}^\perp$ with 2 or more negative eigenvalues.

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian traceless matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues?

The family of $5d - 7$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^2 - 1) - (5d - 7) = (d - 2)(d - 3)$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done: CDJJKSZ Theorem

The basic idea is to construct a basis of $\mathbf{A}^\perp$ along anti-diagonals. An example in the $d = 5$ case is:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & i \\ \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & i & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & i \\ \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \end{bmatrix}$$

How It's Done: CDJJKSZ Theorem

The basic idea is to construct a basis of $\mathbf{A}^\perp$ along anti-diagonals. An example in the $d = 5$ case is:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & i \\ \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & i & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \\ -i & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & i \\ \cdot & \cdot & \cdot & i & \cdot \\ \cdot & \cdot & -i & \cdot & \cdot \\ \cdot & -i & \cdot & \cdot & \cdot \end{bmatrix}$$

Introduction to Process Tomography

Process tomography deals with trying to reconstruct a mathematical description of an unknown quantum process Φ (i.e., completely positive, trace-preserving map).

The approach: have Φ act on part of a given input state ξ and then measure the output state with respect to the observable A.

Performing the above measurement gives us information about $\mathrm{Tr}(A(id \otimes \Phi)(\xi))$.

Introduction to Process Tomography

Process tomography deals with trying to reconstruct a mathematical description of an unknown quantum process Φ (i.e., completely positive, trace-preserving map).

The approach: have Φ act on part of a given input state ξ and then measure the output state with respect to the observable A.

Performing the above measurement gives us information about $\mathrm{Tr}(A(id \otimes \Phi)(\xi))$.

Introduction to Process Tomography

Process tomography deals with trying to reconstruct a mathematical description of an unknown quantum process Φ (i.e., completely positive, trace-preserving map).

The approach: have Φ act on part of a given input state ξ and then measure the output state with respect to the observable A.

Performing the above measurement gives us information about $\mathrm{Tr}(A(id \otimes \Phi)(\xi))$.

Introduction to Process Tomography

To get full information about Φ, we repeat this procedure for many different pairs $(\xi_1, A_1), \ldots, (\xi_m, A_m)$.

Fact: For all $1 \leq k \leq m$, we can choose $\xi_k = |\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle := \sum_i |i\rangle \otimes |i\rangle$.

Then for all k we have $(id \otimes \Phi)(\xi_k) = C_\Phi$, the "Choi matrix", of Φ.

Thus measuring the pair (ξ_k, A_k) gives us information about $\mathrm{Tr}(A_k C_\Phi)$ (i.e., we're just doing state tomography on the Choi matrix of Φ).

Introduction to Process Tomography

To get full information about Φ, we repeat this procedure for many different pairs $(\xi_1, A_1), \ldots, (\xi_m, A_m)$.

Fact: For all $1 \leq k \leq m$, we can choose $\xi_k = |\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle := \sum_i |i\rangle \otimes |i\rangle$.

Then for all k we have $(id \otimes \Phi)(\xi_k) = C_\Phi$, the "Choi matrix", of Φ.

Thus measuring the pair (ξ_k, A_k) gives us information about $\mathrm{Tr}(A_k C_\Phi)$ (i.e., we're just doing state tomography on the Choi matrix of Φ).

Introduction to Process Tomography

To get full information about Φ, we repeat this procedure for many different pairs $(\xi_1, A_1), \ldots, (\xi_m, A_m)$.

Fact: For all $1 \leq k \leq m$, we can choose $\xi_k = |\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle := \sum_i |i\rangle \otimes |i\rangle$.

Then for all k we have $(id \otimes \Phi)(\xi_k) = C_\Phi$, the "Choi matrix", of Φ.

Thus measuring the pair (ξ_k, A_k) gives us information about $\mathrm{Tr}(A_k C_\Phi)$ (i.e., we're just doing state tomography on the Choi matrix of Φ).

Introduction to Process Tomography

To get full information about Φ, we repeat this procedure for many different pairs $(\xi_1, A_1), \ldots, (\xi_m, A_m)$.

Fact: For all $1 \leq k \leq m$, we can choose $\xi_k = |\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle := \sum_i |i\rangle \otimes |i\rangle$.

Then for all k we have $(id \otimes \Phi)(\xi_k) = C_\Phi$, the "Choi matrix", of Φ.

Thus measuring the pair (ξ_k, A_k) gives us information about $\mathrm{Tr}(A_k C_\Phi)$ (i.e., we're just doing state tomography on the Choi matrix of Φ).

Side Note: Interactive Observables

Every measurement pair (ξ, A) (even when ξ is not the maximally entangled state) can be represented by a single **interactive observable** H that satisfies:

$$\mathrm{Tr}(HC_\Phi) = \mathrm{Tr}(A(id \otimes \Phi)(\xi)).$$

Conversely, given a Hermitian operator H that is "small enough", we can find a corresponding measurement pair (ξ, A).

Side Note: Interactive Observables

Every measurement pair (ξ, A) (even when ξ is not the maximally entangled state) can be represented by a single **interactive observable** H that satisfies:

$$\mathrm{Tr}(HC_\Phi) = \mathrm{Tr}(A(id \otimes \Phi)(\xi)).$$

Conversely, given a Hermitian operator H that is "small enough", we can find a corresponding measurement pair (ξ, A).

Tomography of Unitary Channels

Since the set of quantum channels from M_A to M_B spans an affine space of dimension $d^4 - d^2$, we can't expect to reconstruct arbitrary channels with fewer than $d^4 - d^2$ measurement operators $\{A_k\}$.

Question

What if Φ is a unitary channel (i.e., of the form $\Phi(\rho) = U^\dagger \rho U$ for some fixed unitary operator U)?

Fact: $\Phi : M_A \to M_B$ is a unitary channel if and only if C_Φ is a pure state satisfying $\mathrm{Tr}_B(C_\Phi) = I_A$.

Tomography of Unitary Channels

Since the set of quantum channels from M_A to M_B spans an affine space of dimension $d^4 - d^2$, we can't expect to reconstruct arbitrary channels with fewer than $d^4 - d^2$ measurement operators $\{A_k\}$.

Question

What if Φ is a unitary channel (i.e., of the form $\Phi(\rho) = U^\dagger \rho U$ for some fixed unitary operator U)?

Fact: $\Phi : M_A \to M_B$ is a unitary channel if and only if C_Φ is a pure state satisfying $\mathrm{Tr}_B(C_\Phi) = I_A$.

Tomography of Unitary Channels

Since the set of quantum channels from M_A to M_B spans an affine space of dimension $d^4 - d^2$, we can't expect to reconstruct arbitrary channels with fewer than $d^4 - d^2$ measurement operators $\{A_k\}$.

Question

What if Φ is a unitary channel (i.e., of the form $\Phi(\rho) = U^\dagger \rho U$ for some fixed unitary operator U)?

Fact: $\Phi : M_A \to M_B$ is a unitary channel if and only if C_Φ is a pure state satisfying $\mathrm{Tr}_B(C_\Phi) = I_A$.

Tomography of Unitary Channels

Instead of requiring $O(d^4)$ measurement outcomes, we only require $O(d^2)$ in this case:

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

[3] G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.

Tomography of Unitary Channels

Instead of requiring $O(d^4)$ measurement outcomes, we only require $O(d^2)$ in this case:

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

[3] G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.

Tomography of Unitary Channels

Instead of requiring $O(d^4)$ measurement outcomes, we only require $O(d^2)$ in this case:

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

[3] G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.

How It's Done

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Key idea: Suppose Φ and Ψ are unitary channels such that C_Φ and C_Ψ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(C_\Phi - C_\Psi)\big) = 0 \quad \forall 1 \leq j \leq m.$$

That is, $C_\Phi - C_\Psi \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Key idea: Suppose Φ and Ψ are unitary channels such that C_Φ and C_Ψ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(C_\Phi - C_\Psi)\big) = 0 \quad \forall 1 \leq j \leq m.$$

That is, $C_\Phi - C_\Psi \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done

Theorem

There exists a set of $m = 4d^2 - 2d - 4$ observables for which no two unitary channels have the same measurement results.

Key idea: Suppose Φ and Ψ are unitary channels such that C_Φ and C_Ψ do have the same measurement results when measuring $\mathbf{A} = (A_1, A_2, \ldots, A_m)$. Then

$$\mathrm{Tr}\big(A_j(C_\Phi - C_\Psi)\big) = 0 \quad \forall\, 1 \leq j \leq m.$$

That is, $C_\Phi - C_\Psi \in \mathbf{A}^\perp$, the orthogonal complement of the span of $A_1, \ldots, A_m$.

How It's Done

Notice that $C_\Phi - C_\Psi$ always has rank 2, and furthermore $\mathrm{Tr}_B(C_\Phi - C_\Psi) = 0_A$. This leads to the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$ and $\mathrm{Tr}_B(X) = 0_A$?

The family of $4d^2 - 2d - 4$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^4 - d^2) - (4d^2 - 2d - 4) = d^4 - 5d^2 + 2d + 4$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done

Notice that $C_\Phi - C_\Psi$ always has rank 2, and furthermore $\mathrm{Tr}_B(C_\Phi - C_\Psi) = 0_A$. This leads to the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$ and $\mathrm{Tr}_B(X) = 0_A$?

The family of $4d^2 - 2d - 4$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^4 - d^2) - (4d^2 - 2d - 4) = d^4 - 5d^2 + 2d + 4$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done

Notice that $C_\Phi - C_\Psi$ always has rank 2, and furthermore $\mathrm{Tr}_B(C_\Phi - C_\Psi) = 0_A$. This leads to the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$ and $\mathrm{Tr}_B(X) = 0_A$?

The family of $4d^2 - 2d - 4$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^4 - d^2) - (4d^2 - 2d - 4) = d^4 - 5d^2 + 2d + 4$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done

Notice that $C_\Phi - C_\Psi$ always has rank 2, and furthermore $\mathrm{Tr}_B(C_\Phi - C_\Psi) = 0_A$. This leads to the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has $\mathrm{rank}(X) \geq 3$ and $\mathrm{Tr}_B(X) = 0_A$?

The family of $4d^2 - 2d - 4$ observables mentioned earlier corresponds to a subspace $\mathbf{A}^\perp$ of dimension $(d^4 - d^2) - (4d^2 - 2d - 4) = d^4 - 5d^2 + 2d + 4$.

How is this subspace $\mathbf{A}^\perp$ constructed?

How It's Done

We take the HMW construction via matrix diagonals and tweak it so that the **partial trace** equals 0. So the matrices look something like this in the $d = 3$ case:

$$\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\
\hline
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -2 \\
\hline
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot
\end{array}\right]$$

How It's Done

We take the HMW construction via matrix diagonals and tweak it so that the **partial trace** equals 0. So the matrices look something like this in the $d = 3$ case:

$$\left[\begin{array}{ccc|ccc|ccc} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ \hline 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -2 \\ \hline \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \end{array}\right]$$

How It's Done

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

Key idea: Again, the idea is simply to adapt the pure state construction to this new setting. That is, we ask the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues and $\mathrm{Tr}_B(X) = 0_A$?

How It's Done

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

Key idea: Again, the idea is simply to adapt the pure state construction to this new setting. That is, we ask the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues and $\mathrm{Tr}_B(X) = 0_A$?

How It's Done

Theorem

There exists a set of $m = 5d^2 - 3d - 4$ observables for which every unitary channel is such that no other channel (unitary or otherwise) has the same measurement results.

Key idea: Again, the idea is simply to adapt the pure state construction to this new setting. That is, we ask the following question:

Question

What is the maximal dimension of a subspace $\mathbf{A}^\perp$ of the Hermitian matrices so that every $0 \neq X \in \mathbf{A}^\perp$ has at least 2 negative eigenvalues and $\mathrm{Tr}_B(X) = 0_A$?

How It's Done: Unital–All Channels Case

We take the CDJJKSZ construction via matrix anti-diagonals and tweak it so that the partial trace equals 0. So the matrices look something like this in the $d = 3$ case:

$$\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
-1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

How It's Done: Unital–All Channels Case

We take the CDJJKSZ construction via matrix anti-diagonals and tweak it so that the **partial trace** equals 0. So the matrices look something like this in the $d = 3$ case:

$$\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
-1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

Closing Questions

How well can we do if we restrict the input state ξ so that it is in some set of states that is easy to prepare?

What if we restrict the observables to be in some set that is experimentally-viable (e.g., elements of the Clifford group)?

Can the optimal number of observables be computed for small d?

Thank You!

G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.

Closing Questions

How well can we do if we restrict the input state ξ so that it is in some set of states that is easy to prepare?

What if we restrict the observables to be in some set that is experimentally-viable (e.g., elements of the Clifford group)?

Can the optimal number of observables be computed for small d?

Thank You!

G. Gutoski and N. J. Process tomography for unitary quantum channels.
E-print: arXiv:1309.0840 [quant-ph], 2013.

Closing Questions

How well can we do if we restrict the input state ξ so that it is in some set of states that is easy to prepare?

What if we restrict the observables to be in some set that is experimentally-viable (e.g., elements of the Clifford group)?

Can the optimal number of observables be computed for small d?

Thank You!

G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.

Closing Questions

How well can we do if we restrict the input state ξ so that it is in some set of states that is easy to prepare?

What if we restrict the observables to be in some set that is experimentally-viable (e.g., elements of the Clifford group)?

Can the optimal number of observables be computed for small d?

Thank You!

G. Gutoski and N. J. Process tomography for unitary quantum channels. E-print: arXiv:1309.0840 [quant-ph], 2013.