

Limitations on separable measurements by convex optimization

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Abstract

We prove limitations on LOCC and separable measurements in bipartite state discrimination problems using techniques from convex optimization. Specific results that we prove include: an exact formula for the optimal probability of correctly discriminating any set of either three or four Bell states via LOCC or separable measurements when the parties are given an ancillary partially entangled pair of qubits; an easily checkable characterization of when an unextendable product set is perfectly discriminated by separable measurements, along with the first known example of an unextendable product set that cannot be perfectly discriminated by separable measurements; and an optimal bound on the success probability for any LOCC or separable measurement for the recently proposed state discrimination problem of Yu, Duan, and Ying.

1 Introduction

The paradigm of *local operations and classical communication*, or LOCC for short, is fundamental within the theory of quantum information. A protocol involving two or more individuals is said to be an LOCC protocol when it may be implemented by means of classical communication among the individuals, along with arbitrary quantum operations performed locally. This paradigm serves as a foundation through which properties of entanglement may be studied, particularly those connected with the notion of entanglement as a resource for information processing.

State discrimination problems

One basic problem concerning the LOCC paradigm that has been studied in depth regards the discrimination of sets of bipartite (or multipartite) states by means of measurements that can be implemented by LOCC protocols. In the most typically considered variant of this problem, one first specifies an ensemble of states

$$\{(p_1, \rho_1), \dots, (p_N, \rho_N)\}, \quad (1)$$

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where N is a positive integer, (p_1, \dots, p_N) is a probability vector, and ρ_1, \dots, ρ_N are density operators representing quantum states of systems shared between two separate individuals: Alice and Bob. With respect to the probability vector (p_1, \dots, p_N) , an index $k \in \{1, \dots, N\}$ is selected at random, and Alice and Bob are given the quantum state ρ_k for the selected index k . Their goal is to determine the index k of the given state ρ_k by means of an LOCC measurement.

In most prior works on this problem, the probability vector (p_1, \dots, p_N) has been taken to be $(1/N, \dots, 1/N)$, representing a uniform probability distribution, and the states ρ_1, \dots, ρ_N have been taken to be pure and orthogonal, so that a global measurement can trivially discriminate them with certainty. Many examples are known of specific choices of pure, orthogonal states ρ_1, \dots, ρ_N for which a perfect discrimination is not possible through LOCC measurements. Some of these examples, along with other general results concerning this problem, may be found in [BGK11, BN13, BW09, BDF⁺99, CL03, DFX09, Fan04, GKR⁺01, GKRS04, HMM⁺06, HSSH03, Nat05, Nat13, WH02, WSHV00, Wat05, YDY12, YDY14].

As perhaps the simplest example of an instance of this problem where a perfect LOCC discrimination is not possible, one has that four Bell states,

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle, & |\phi_2\rangle &= \frac{1}{\sqrt{2}}|0\rangle|0\rangle - \frac{1}{\sqrt{2}}|1\rangle|1\rangle, \\ |\phi_3\rangle &= \frac{1}{\sqrt{2}}|0\rangle|1\rangle + \frac{1}{\sqrt{2}}|1\rangle|0\rangle, & |\phi_4\rangle &= \frac{1}{\sqrt{2}}|0\rangle|1\rangle - \frac{1}{\sqrt{2}}|1\rangle|0\rangle, \end{aligned} \quad (2)$$

cannot be perfectly discriminated by LOCC measurements [GKR⁺01]. More precisely, if one takes $N = 4$, $\rho_k = |\phi_k\rangle\langle\phi_k|$, and $p_k = 1/4$ for each $k \in \{1, \dots, 4\}$ in the above problem, it holds that the optimal probability with which Alice and Bob can correctly determine the chosen index $k \in \{1, \dots, 4\}$, assuming that they are restricted to local operations and classical communication, is $1/2$. The fact that Alice and Bob can achieve a success probability of $1/2$ is straightforward: if they both measure their qubit with respect to the standard basis, compare the results through classical communication, and answer $k = 1$ if the measurements agree and $k = 3$ if the measurements disagree, they will be correct with probability $1/2$. The fact that they cannot achieve a probability of correctness larger than $1/2$ follows from a result of Nathanson [Nat05] stating that N equiprobable, *maximally entangled*, bipartite states having local dimension n can be discriminated correctly with probability at most n/N . Consequently, even in the variant of this example in which $N = 3$, $\rho_k = |\phi_k\rangle\langle\phi_k|$, and $p_k = 1/3$ for each $k \in \{1, 2, 3\}$, Alice and Bob cannot achieve a probability of correctness larger than $2/3$, which again is achievable through the simple protocol described above. (More generally, if the states $\rho_1, \rho_2, \rho_3, \rho_4$ are given with probabilities $p_1 \geq p_2 \geq p_3 \geq p_4$, the optimal probability of a correct discrimination by an LOCC measurement is $p_1 + p_2$ [BN13].)

Among the other known examples of collections of orthogonal pure states that cannot be perfectly discriminated by LOCC protocols, the so-called *domino state* example of [BDF⁺99] is noteworthy. In this example, the local dimension of the states is 3, and one takes $N = 9$, $p_1 = \dots = p_9 = 1/9$, and

$$\begin{aligned} |\phi_1\rangle &= |1\rangle|1\rangle, \\ |\phi_2\rangle &= |0\rangle \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \right), \quad |\phi_3\rangle = |2\rangle \left(\frac{|1\rangle+|2\rangle}{\sqrt{2}} \right), \quad |\phi_4\rangle = \left(\frac{|1\rangle+|2\rangle}{\sqrt{2}} \right) |0\rangle, \quad |\phi_5\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} \right) |2\rangle, \\ |\phi_6\rangle &= |0\rangle \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \right), \quad |\phi_7\rangle = |2\rangle \left(\frac{|1\rangle-|2\rangle}{\sqrt{2}} \right), \quad |\phi_8\rangle = \left(\frac{|1\rangle-|2\rangle}{\sqrt{2}} \right) |0\rangle, \quad |\phi_9\rangle = \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}} \right) |2\rangle. \end{aligned} \quad (3)$$

A rather complicated argument demonstrates that this collection cannot be discriminated with probability greater than $1 - \varepsilon$ for some choice of a positive real number ε . (A somewhat simplified proof appears in [CLMO13], where this fact is proved for $\varepsilon = 1.9 \times 10^{-8}$.) The particular relevance of this example lies in the fact that all of these states are product states, demonstrating that entanglement is not a requisite for a set of orthogonal pure states to fail to be perfectly discriminated by any LOCC measurement. Complementary to this observation, a fundamental result of Walgate, Short, Hardy, and Vedral [WSHV00] states that any two orthogonal pure states (whether entangled or not) can always be perfectly discriminated by an LOCC measurement.

Separable measurements

The set of measurements that can be implemented through LOCC has an apparently complex mathematical structure—no tractable characterization of this set is known, representing a clear obstacle to a better understanding of the limitations of LOCC measurements. For example, given a collection of measurement operators $\{P_1, \dots, P_N\}$ describing a measurement on a bipartite system, the determination of whether or not this collection describes an LOCC measurement, or is closely approximated by an LOCC measurement, is not known to be a computationally decidable problem.

For this reason, the state discrimination problem described above is sometimes considered for more tractable classes of measurements that approximate, in some sense, the LOCC measurements. The class of *separable measurements* represents one commonly studied approximation in this category. Let us assume hereafter that $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ are complex Euclidean spaces (or, equivalently, finite-dimensional complex Hilbert spaces) representing the local systems of Alice and Bob, respectively, in a state discrimination problem. A positive semidefinite operator $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is said to be *separable* if it is possible to write

$$P = \sum_{k=1}^M Q_k \otimes R_k \tag{4}$$

for some choice of a positive integer M and positive semidefinite operators $Q_1, \dots, Q_M \in \text{Pos}(\mathcal{X})$ and $R_1, \dots, R_M \in \text{Pos}(\mathcal{Y})$; and a measurement $\{P_1, \dots, P_N\}$ on $\mathcal{X} \otimes \mathcal{Y}$ is said to be a *separable measurement* if it is the case that each measurement operator P_k is separable. (Here, and elsewhere in the paper, $\text{Pos}(\mathcal{X})$, $\text{Pos}(\mathcal{Y})$, and $\text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ denote the sets of positive semidefinite operators acting on \mathcal{X} , \mathcal{Y} , and $\mathcal{X} \otimes \mathcal{Y}$, respectively.)

It is straightforward to prove that every measurement that can be implemented through LOCC is necessarily a separable measurement, from which it follows that any limitation proved to hold for every separable measurement must also hold for every LOCC measurement. There are, however, separable measurements that cannot be implemented by LOCC protocols; a measurement with respect to the orthonormal basis given by the domino state example (3) is the archetypal example.

Many of the known results that establish limitations on LOCC measurements for state discrimination tasks hold more generally for separable measurements, and may be proved within this somewhat simpler setting. For instance, the result of Nathanson mentioned earlier establishes that, in the bipartite setting in which $n = m$ (i.e., Alice and Bob's local systems are represented by $\mathcal{X} = \mathcal{Y} = \mathbb{C}^n$), one has that N equiprobable, maximally entangled states cannot be discriminated with success probability exceeding n/N by any separable measurement.

PPT measurements

Another class of measurements, representing a further relaxation of the LOCC condition, is the class of PPT measurements. A positive semidefinite operator $P \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is a *PPT* (short for *positive partial transpose*) operator if it holds that

$$T_{\mathcal{X}}(P) \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y}), \quad (5)$$

where $T_{\mathcal{X}} : L(\mathcal{X} \otimes \mathcal{Y}) \rightarrow L(\mathcal{X} \otimes \mathcal{Y})$ is the linear mapping representing partial transposition with respect to the standard basis $\{|0\rangle, \dots, |n-1\rangle\}$ of \mathcal{X} . A measurement is a *PPT measurement* if it is represented by a collection of PPT measurement operators $\{P_1, \dots, P_N\}$. Every separable operator is a PPT operator, so that every separable measurement (and therefore every LOCC measurement) is a PPT measurement as well.

The primary appeal of the set of PPT measurements is its mathematical simplicity. In particular, the PPT condition is represented by linear and positive semidefinite constraints, which allows for an optimization over the collection of PPT measurements to be represented by a semidefinite program. By the duality theory of semidefinite programs, one may obtain upper bounds on the success probability of any PPT measurement (and therefore any LOCC or separable measurement) for a given state discrimination problem; this may be done by simply exhibiting a feasible solution to the dual problem of the semidefinite program representing the measurement optimization for this set of states. The downside of this approach is that the set of PPT measurements is a coarse approximation to the set of LOCC measurements, so the method will fail to prove strong impossibility results for LOCC measurements in many cases.

The approach described above, in which PPT measurements are represented by semidefinite programs, was taken in [Cos13]. There, it was shown that the state discrimination problem of Yu, Duan, and Ying [YDY12], to discriminate the four maximally entangled states

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{2}|0\rangle|0\rangle + \frac{1}{2}|1\rangle|1\rangle + \frac{1}{2}|2\rangle|2\rangle + \frac{1}{2}|3\rangle|3\rangle \\ |\phi_2\rangle &= \frac{1}{2}|0\rangle|3\rangle + \frac{1}{2}|1\rangle|2\rangle + \frac{1}{2}|2\rangle|1\rangle + \frac{1}{2}|3\rangle|0\rangle \\ |\phi_3\rangle &= \frac{1}{2}|0\rangle|3\rangle + \frac{1}{2}|1\rangle|2\rangle - \frac{1}{2}|2\rangle|1\rangle - \frac{1}{2}|3\rangle|0\rangle \\ |\phi_4\rangle &= \frac{1}{2}|0\rangle|1\rangle + \frac{1}{2}|1\rangle|0\rangle - \frac{1}{2}|2\rangle|3\rangle - \frac{1}{2}|3\rangle|2\rangle \end{aligned} \quad (6)$$

by means of a PPT measurement, assuming the states are given with equal probability, is at most $7/8$. (Yu, Duan, and Ying proved that this set cannot be perfectly discriminated by any PPT measurement through a different argument, giving the first example where $N = n$ maximally entangled states having local dimension n cannot be perfectly discriminated by LOCC measurements. Examples of sets of $N < n$ states that cannot be discriminated without error by PPT measurements were later given in [CR14].)

Overview of results

This paper proves several new results concerning state discrimination by LOCC, separable, and PPT measurements using techniques based on convex optimization. Our results are primarily focused on separable measurements, and are mostly based on the paradigm of *cone programming*,

which is a generalization of linear programming and semidefinite programming that allows for optimizations over general closed, convex cones. We also obtain new results based on linear programming and semidefinite programming. The notion of *duality*, shared by linear programs, semidefinite programs, and cone programs, plays a central role in our results. The following specific results are among those we prove:

- We obtain an exact formula for the optimal probability of correctly discriminating any set of either three or four Bell states via separable measurements, when the parties are given an ancillary partially entangled pair of qubits. In particular, it is proved that this ancillary pair of qubits must be maximally entangled in order for three Bell states to be perfectly discriminated by separable (or LOCC) measurements, which answers an open question of [YDY14].
- We provide an easily checkable characterization of when an unextendable product set is perfectly discriminated by separable measurements, and we use this characterization to present an example of an unextendable product set in $\mathcal{X} \otimes \mathcal{Y}$, for $\mathcal{X} = \mathcal{Y} = \mathbb{C}^4$, that is not perfectly discriminated by separable measurements. This resolves an open question raised in [DFXY09]. We also show that every unextendable product set together with one extra pure state orthogonal to every member of the unextendable product set is not perfectly discriminated by separable measurements.
- It is proved that the maximum success probability for any separable measurement in the state discrimination problem of Yu, Duan, and Ying specified above is $3/4$. This bound is easily seen to be achievable by an LOCC measurement, implying that it is the optimal success probability of an LOCC measurement for this problem. The upper-bound is closely connected to the positive maps of Breuer and Hall [Bre06, Hal06].

2 A cone program for optimizing over separable measurements

A *cone program* (also known as a *conic program*) expresses the maximization of a linear function over the intersection of an affine subspace and a closed convex cone in a finite-dimensional real inner product space [BV04]. Linear programming and semidefinite programming are special cases of cone programming: in linear programming, the closed convex cone over which the optimization occurs is the positive orthant in \mathbb{R}^n , while in semidefinite programming the optimization is over the cone $\text{Pos}(\mathbb{C}^n)$ of positive semidefinite operators on \mathbb{C}^n . In the case of semidefinite programming, the finite-dimensional real inner product space is the real vector space $\text{Herm}(\mathbb{C}^n)$ of Hermitian operators on \mathbb{C}^n , equipped with the Hilbert-Schmidt inner product: $\langle X, Y \rangle = \text{Tr}(XY)$. One may also consider semidefinite programming over real positive semidefinite operators.

Cone programming

For the purposes of the present paper, it is sufficient to consider only cone programs defined over spaces of Hermitian operators (with the Hilbert-Schmidt inner product). In particular, let \mathcal{Z} and \mathcal{W} be complex Euclidean spaces, let $\text{Herm}(\mathcal{Z})$ and $\text{Herm}(\mathcal{W})$ denote the sets of Hermitian operators acting on \mathcal{Z} and \mathcal{W} , respectively, and let $\mathcal{K} \subseteq \text{Herm}(\mathcal{Z})$ be a closed, convex cone. For any choice of a linear map $\Phi : \text{Herm}(\mathcal{Z}) \rightarrow \text{Herm}(\mathcal{W})$ and Hermitian operators $A \in \text{Herm}(\mathcal{Z})$ and $B \in \text{Herm}(\mathcal{W})$, one has a cone program, which is represented by a pair of optimization problems:

| Primal problem | Dual problem |
|----------------------------------|--|
| maximize: $\langle A, X \rangle$ | minimize: $\langle B, Y \rangle$ |
| subject to: $\Phi(X) = B,$ | subject to: $\Phi^*(Y) - A \in \mathcal{K}^*,$ |
| $X \in \mathcal{K}.$ | $Y \in \text{Herm}(\mathcal{W}).$ |

Here, \mathcal{K}^* denotes the *dual cone* to \mathcal{K} , defined as

$$\mathcal{K}^* = \{Y \in \text{Herm}(\mathcal{Z}) : \langle X, Y \rangle \geq 0 \text{ for all } X \in \mathcal{K}\} \quad (7)$$

and $\Phi^* : \text{Herm}(\mathcal{W}) \rightarrow \text{Herm}(\mathcal{Z})$ is the adjoint mapping to Φ , which is uniquely determined by the equation $\langle Y, \Phi(X) \rangle = \langle \Phi^*(Y), X \rangle$ holding for all $X \in \text{Herm}(\mathcal{Z})$ and $Y \in \text{Herm}(\mathcal{W})$.

With the optimization problems above in mind, one defines the *feasible sets* \mathcal{A} and \mathcal{B} of the primal and dual problems as

$$\mathcal{A} = \{X \in \mathcal{K} : \Phi(X) = B\} \quad \text{and} \quad \mathcal{B} = \{Y \in \text{Herm}(\mathcal{W}) : \Phi^*(Y) - A \in \mathcal{K}^*\}. \quad (8)$$

One says that the associated cone program is *primal feasible* if $\mathcal{A} \neq \emptyset$, and is *dual feasible* if $\mathcal{B} \neq \emptyset$. The function $X \mapsto \langle A, X \rangle$ from $\text{Herm}(\mathcal{Z})$ to \mathbb{R} is called the *primal objective function*, and the function $Y \mapsto \langle B, Y \rangle$ from $\text{Herm}(\mathcal{W})$ to \mathbb{R} is called the *dual objective function*. The *optimal values* associated with the primal and dual problems are defined as

$$\alpha = \sup\{\langle A, X \rangle : X \in \mathcal{A}\} \quad \text{and} \quad \beta = \inf\{\langle B, Y \rangle : Y \in \mathcal{B}\}, \quad (9)$$

respectively. (It is conventional to interpret that $\alpha = -\infty$ when $\mathcal{A} = \emptyset$ and $\beta = \infty$ when $\mathcal{B} = \emptyset$.) The property of *weak duality*, which holds for all cone programs, is that the primal optimum can never exceed the dual optimum.

Proposition 1 (Weak duality for cone programs). *For any choice of complex Euclidean spaces \mathcal{Z} and \mathcal{W} , a closed, convex cone $\mathcal{K} \subseteq \text{Herm}(\mathcal{Z})$, Hermitian operators $A \in \text{Herm}(\mathcal{Z})$ and $B \in \text{Herm}(\mathcal{W})$, and a linear map $\Phi : \text{Herm}(\mathcal{Z}) \rightarrow \text{Herm}(\mathcal{W})$, it holds that $\alpha \leq \beta$, for α and β as defined in (9).*

Proof. The proposition is trivial in case $\mathcal{A} = \emptyset$ (which implies that $\alpha = -\infty$) or $\mathcal{B} = \emptyset$ (which implies that $\beta = \infty$), so we will restrict our attention to the case that both \mathcal{A} and \mathcal{B} are nonempty. For any choice of $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, one must have $X \in \mathcal{K}$ and $\Phi^*(Y) - A \in \mathcal{K}^*$, and therefore $\langle \Phi^*(Y) - A, X \rangle \geq 0$. It follows that

$$\langle A, X \rangle = \langle \Phi^*(Y), X \rangle - \langle \Phi^*(Y) - A, X \rangle \leq \langle Y, \Phi(X) \rangle = \langle B, Y \rangle. \quad (10)$$

Taking the supremum over all $X \in \mathcal{A}$ and the infimum over all $Y \in \mathcal{B}$ establishes that $\alpha \leq \beta$. \square

Weak duality implies that every dual feasible operator $Y \in \mathcal{B}$ provides an upper bound of $\langle B, Y \rangle$ on the value $\langle A, X \rangle$ that is achievable over all choices of a primal feasible $X \in \mathcal{A}$, and likewise every primal feasible operator $X \in \mathcal{A}$ provides a lower bound of $\langle A, X \rangle$ on the value $\langle B, Y \rangle$ that is achievable over all choices of a dual feasible solution $Y \in \mathcal{B}$. In other words, it holds that $\langle A, X \rangle \leq \alpha \leq \beta \leq \langle B, Y \rangle$, for every $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

There are simple conditions under which the primal and dual optimal values will in fact be equal, which is a situation known as *strong duality*. Although the cone programs considered in this paper do indeed possess this stronger notion of duality, it is not needed for any of our results.

Optimizing over separable measurements

Let us now return to the state discrimination problem. Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ be complex Euclidean spaces corresponding to quantum systems held by Alice and Bob, respectively, and let $\{\rho_1, \dots, \rho_N\} \subset \mathcal{D}(\mathcal{X} \otimes \mathcal{Y})$ be a set of density operators, representing quantum states of Alice and Bob's shared systems. Alice and Bob are given a state ρ_k , for some index $k \in \{1, \dots, N\}$ drawn according to a probability distribution (p_1, \dots, p_N) , as was described above in the introduction. Their goal is to maximize the probability that they correctly identify the chosen index k , assuming that they have complete knowledge of the set $\{\rho_1, \dots, \rho_N\}$ and the probability distribution (p_1, \dots, p_N) . Our focus is on the situation in which they do this by means of a separable measurement $\{P_1, \dots, P_N\}$.

Hereafter, let us denote the set of all separable operators acting on the space $\mathcal{X} \otimes \mathcal{Y}$ by $\text{Sep}(\mathcal{X} : \mathcal{Y})$. One may observe that $\text{Sep}(\mathcal{X} : \mathcal{Y})$ is a closed, convex cone, which will allow an optimization over separable measurements $\{P_1, \dots, P_N\}$ to be expressed as a cone program. The dual cone to $\text{Sep}(\mathcal{X} : \mathcal{Y})$, which is commonly known as the set of *block-positive operators*, is defined as

$$\text{Sep}^*(\mathcal{X} : \mathcal{Y}) = \{H \in \text{Herm}(\mathcal{X} \otimes \mathcal{Y}) : \langle P, H \rangle \geq 0 \text{ for every } P \in \text{Sep}(\mathcal{X} : \mathcal{Y})\}. \quad (11)$$

There are several equivalent characterizations of this set. For instance, one has

$$\text{Sep}^*(\mathcal{X} : \mathcal{Y}) = \{H \in \text{Herm}(\mathcal{X} \otimes \mathcal{Y}) : (\mathbb{1}_{\mathcal{X}} \otimes y^*)H(\mathbb{1}_{\mathcal{X}} \otimes y) \in \text{Pos}(\mathcal{X}) \text{ for every } y \in \mathcal{Y}\}. \quad (12)$$

Alternatively, block-positive operators can be characterized as representations of *positive* linear maps, via the Choi representation. That is, for any linear map $\Phi : \mathcal{L}(\mathcal{Y}) \rightarrow \mathcal{L}(\mathcal{X})$ mapping arbitrary linear operators on \mathcal{Y} to linear operators on \mathcal{X} , the following two properties are equivalent:

- (a) For every positive semidefinite operator $Y \in \text{Pos}(\mathcal{Y})$, it holds that $\Phi(Y) \in \text{Pos}(\mathcal{X})$.
- (b) The Choi operator

$$J(\Phi) = \sum_{0 \leq j, k < m} \Phi(|j\rangle\langle k|) \otimes |j\rangle\langle k| \quad (13)$$

of Φ satisfies $J(\Phi) \in \text{Sep}^*(\mathcal{X} : \mathcal{Y})$.

We now observe that the following cone program represents the optimal value of a correct state discrimination in the setting under consideration. The primal problem is as follows:

$$\begin{aligned} \text{maximize: } & p_1 \langle \rho_1, P_1 \rangle + \dots + p_N \langle \rho_N, P_N \rangle \\ \text{subject to: } & P_1 + \dots + P_N = \mathbb{1}_{\mathcal{X} \otimes \mathcal{Y}} \\ & P_k \in \text{Sep}(\mathcal{X} : \mathcal{Y}) \quad (\text{for each } k = 1, \dots, N), \end{aligned} \quad (14)$$

and the dual problem is as follows:

$$\begin{aligned} \text{minimize: } & \text{Tr}(H) \\ \text{subject to: } & H - p_k \rho_k \in \text{Sep}^*(\mathcal{X} : \mathcal{Y}) \quad (\text{for each } k = 1, \dots, N) \\ & H \in \text{Herm}(\mathcal{X} \otimes \mathcal{Y}). \end{aligned} \quad (15)$$

If one is to formally specify this problem according to the general form for cone programs presented above, the operators P_1, \dots, P_N may be represented as a block matrix of the form

$$X = \begin{pmatrix} P_1 & \dots & \cdot \\ \vdots & \ddots & \vdots \\ \cdot & \dots & P_N \end{pmatrix} \in \text{Herm}((\mathcal{X} \otimes \mathcal{Y}) \oplus \dots \oplus (\mathcal{X} \otimes \mathcal{Y})) \quad (16)$$

with the off-diagonal blocks being left unspecified. The cone \mathcal{K} is taken to be the cone of operators of this form for which each P_k is separable, and the mapping Φ and operators A and B are chosen in the natural way:

$$A = \begin{pmatrix} p_1 \rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_N \rho_N \end{pmatrix}, \quad B = \mathbb{1}, \quad \text{and} \quad \Phi \begin{pmatrix} P_1 & \cdots & \cdot \\ \vdots & \ddots & \vdots \\ \cdot & \cdots & P_N \end{pmatrix} = P_1 + \cdots + P_N. \quad (17)$$

One may verify that the dual problem is as claimed by a straightforward computation.

By weak duality for cone programs, an upper bound on the probability with which a separable measurement can discriminate the states ρ_1, \dots, ρ_N given with probabilities p_1, \dots, p_N is given by every dual feasible solution to this cone program: for any Hermitian operator $H \in \text{Herm}(\mathcal{X} \otimes \mathcal{Y})$ for which $H - p_k \rho_k$ is block positive for every $k = 1, \dots, N$, the probability of a correct discrimination is upper-bounded by $\text{Tr}(H)$.

3 The entanglement cost of discriminating Bell states

As explained in the introduction, three Bell states given with uniform probabilities can be discriminated by separable measurements with success probability at most $2/3$, while four can be discriminated with success probability at most $1/2$. These bounds can be obtained by a fairly trivial selection of LOCC measurements, and can be shown to hold even for PPT measurements.

In this section, we study state discrimination problems for sets of three or four Bell states, by LOCC, separable, and PPT measurements, with the assistance of an entangled pair of qubits. In particular, we will assume that Alice and Bob aim to discriminate a set of Bell states given that they share the additional resource state

$$|\tau_\varepsilon\rangle = \sqrt{\frac{1+\varepsilon}{2}} |0\rangle|0\rangle + \sqrt{\frac{1-\varepsilon}{2}} |1\rangle|1\rangle, \quad (18)$$

for some choice of $\varepsilon \in [0, 1]$. The parameter ε quantifies the amount of entanglement in the state $|\tau_\varepsilon\rangle$. Up to local unitaries, this family of states represents every pure state of two qubits.

The entanglement cost of quantum operations and measurements, within the paradigm of LOCC, has been considered previously. For instance, [S08] studied the entanglement cost of perfectly discriminating elements of unextendable product sets by LOCC measurements, [BBKW09] and [BRW10] considered the entanglement cost of measurements and established lower bounds on the amount of entanglement necessary for distinguishing complete orthonormal bases of two qubits, and [YDY14] considered the entanglement cost of state discrimination problems by PPT and separable measurements.

Using the cone programming method discussed in the previous section, we obtain exact expressions for the optimal probability with which any set of three or four Bell states can be discriminated with the assistance of the state (18) by separable measurements (which match the probabilities obtained by LOCC measurements in all cases). This answers an open question raised in [YDY14].

Discriminating three Bell states

Notice that the state $|\tau_1\rangle = |0\rangle|0\rangle$ is a product state and it does not aid the two parties in discriminating any set of Bell states, so the probability of success for $\varepsilon = 1$ is still at most $2/3$ for a

set of three Bell states. If $\varepsilon = 0$, then Alice and Bob can use teleportation to perfectly discriminate all four Bell states perfectly by LOCC measurements, and therefore the same is true for any three Bell states. It was proved in [YDY14] that PPT measurements can perfectly discriminate any set of three Bell states using the resource state (18) if and only if $\varepsilon \leq 1/3$.

Here we show that a maximally entangled state ($\varepsilon = 0$) is required to perfectly discriminate any set of three Bell states using separable measurements, and more generally we obtain an expression for the optimal probability of a correct discrimination for all values of ε . Because the permutations of Bell states induced by local unitaries is transitive, there is no loss of generality in fixing the three Bell states to be discriminated to be $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ (as defined in (2)).

Theorem 2. *Let $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{C}^2$, define $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \otimes \mathcal{Y}_2$, and let $\varepsilon \in [0, 1]$ be chosen arbitrarily. For any separable measurement $\{P_1, P_2, P_3\} \subset \text{Sep}(\mathcal{X} : \mathcal{Y})$, the success probability of correctly discriminating the states corresponding to the set*

$$\{|\phi_1\rangle \otimes |\tau_\varepsilon\rangle, |\phi_2\rangle \otimes |\tau_\varepsilon\rangle, |\phi_3\rangle \otimes |\tau_\varepsilon\rangle\} \subset (\mathcal{X}_1 \otimes \mathcal{Y}_1) \otimes (\mathcal{X}_2 \otimes \mathcal{Y}_2), \quad (19)$$

assuming a uniform distribution $p_1 = p_2 = p_3 = 1/3$, is at most

$$\frac{1}{3} \left(2 + \sqrt{1 - \varepsilon^2} \right). \quad (20)$$

To prove this theorem, we require the following lemma. The lemma introduces a family of positive maps that, to our knowledge, has not previously appeared in the literature.

Lemma 3. *Define a linear mapping $\Xi_t : L(\mathbb{C}^2 \oplus \mathbb{C}^2) \rightarrow L(\mathbb{C}^2 \oplus \mathbb{C}^2)$ as*

$$\Xi_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Psi_t(D) + \Phi(D) & \Psi_t(B) + \Phi(C) \\ \Psi_t(C) + \Phi(B) & \Psi_t(A) + \Phi(A) \end{pmatrix} \quad (21)$$

for every $t \in (0, \infty)$ and $A, B, C, D \in L(\mathbb{C}^2)$, where $\Psi_t : L(\mathbb{C}^2) \rightarrow L(\mathbb{C}^2)$ is defined as

$$\Psi_t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} t\alpha & \beta \\ \gamma & t^{-1}\delta \end{pmatrix} \quad (22)$$

and $\Phi : L(\mathbb{C}^2) \rightarrow L(\mathbb{C}^2)$ is defined as

$$\Phi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \quad (23)$$

for every $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. It holds that Ξ_t is a positive map for all $t \in (0, \infty)$.

Proof. It will first be proved that Ξ_1 is positive. For every vector

$$u = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (24)$$

in \mathbb{C}^2 , define a matrix

$$M_u = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}. \quad (25)$$

Straightforward computations reveal that

$$M_u^* M_v = uv^* + \Phi(vu^*) \quad \text{and} \quad M_u^* M_u = \|u\|^2 \mathbf{1} \quad (26)$$

for all $u, v \in \mathbb{C}^2$. It follows that

$$\Xi_1 \begin{pmatrix} uu^* & uv^* \\ vu^* & vv^* \end{pmatrix} = \begin{pmatrix} vv^* + \Phi(vv^*) & uv^* + \Phi(vu^*) \\ vu^* + \Phi(uv^*) & uu^* + \Phi(uu^*) \end{pmatrix} = \begin{pmatrix} \|v\|^2 \mathbb{1} & M_u^* M_v \\ M_v^* M_u & \|u\|^2 \mathbb{1} \end{pmatrix}, \quad (27)$$

which is positive semidefinite by virtue of the fact that $\|M_u^* M_v\| \leq \|M_u\| \|M_v\| = \|u\| \|v\|$. As every element of $\text{Pos}(\mathbb{C}^2 \oplus \mathbb{C}^2)$ can be written as a positive linear combination of matrices of the form

$$\begin{pmatrix} uu^* & uv^* \\ vu^* & vv^* \end{pmatrix}, \quad (28)$$

ranging over all vectors $u, v \in \mathbb{C}^2$, it follows that Ξ_1 is a positive map.

For the general case, observe first that the mapping Ψ_s may be expressed using the Hadamard (or entry-wise) product as

$$\Psi_s \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} s\alpha & \beta \\ \gamma & s^{-1}\delta \end{pmatrix} = \begin{pmatrix} s & 1 \\ 1 & s^{-1} \end{pmatrix} \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (29)$$

for every positive real number $s \in (0, \infty)$. The matrix

$$\begin{pmatrix} s & 1 \\ 1 & s^{-1} \end{pmatrix} \quad (30)$$

is positive semidefinite, from which it follows (by the Schur product theorem) that Ψ_s is a completely positive map. (See, for instance, Theorem 3.7 of [Pau02].) Also note that $\Phi = \Psi_s \Phi \Psi_s$ for every $s \in (0, \infty)$, which implies that

$$\Xi_t = (\mathbb{1}_{\mathbb{L}(\mathbb{C}^2)} \otimes \Psi_s) \Xi_1 (\mathbb{1}_{\mathbb{L}(\mathbb{C}^2)} \otimes \Psi_s) \quad (31)$$

for $s = \sqrt{t}$. This shows that Ξ_t is a composition of positive maps for every positive real number t , and is therefore positive. \square

Proof of Theorem 2. For the cases that $\varepsilon = 0$ and $\varepsilon = 1$, the theorem is known, as was discussed previously, so it will be assumed that $\varepsilon \in (0, 1)$. Define a Hermitian operator

$$H_\varepsilon = \frac{1}{3} \left[\frac{\mathbb{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} \otimes \tau_\varepsilon}{2} + \sqrt{1 - \varepsilon^2} \phi_4 \otimes T_{\mathcal{X}_2}(\phi_4) \right], \quad (32)$$

where $\tau_\varepsilon = |\tau_\varepsilon\rangle\langle\tau_\varepsilon|$, $\phi_4 = |\phi_4\rangle\langle\phi_4|$, and $T_{\mathcal{X}_2}$ denotes partial transposition with respect to the standard basis of \mathcal{X}_2 . It holds that

$$\text{Tr}(H_\varepsilon) = \frac{1}{3} \left(2 + \sqrt{1 - \varepsilon^2} \right), \quad (33)$$

so to complete the proof it suffices to prove that H_ε is a feasible solution to the dual problem (15) for the cone program corresponding to the state discrimination problem being considered.

In order to be more precise about the task at hand, it is helpful to define a unitary operator W , mapping $\mathcal{X}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_1 \otimes \mathcal{Y}_2$ to $\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2$, that corresponds to swapping the second and third subsystems:

$$W(x_1 \otimes x_2 \otimes y_1 \otimes y_2) = x_1 \otimes y_1 \otimes x_2 \otimes y_2, \quad (34)$$

for all vectors $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, $y_1 \in \mathcal{Y}_1$, $y_2 \in \mathcal{Y}_2$. We are concerned with the separability of measurement operators with respect to the bipartition between $\mathcal{X}_1 \otimes \mathcal{X}_2$ and $\mathcal{Y}_1 \otimes \mathcal{Y}_2$, so the dual feasibility of H_ε requires that the operators defined as

$$Q_{k,\varepsilon} = W^* \left(H_\varepsilon - \frac{1}{3} \phi_k \otimes \tau_\varepsilon \right) W \in \text{Herm}(\mathcal{X} \otimes \mathcal{Y}) \quad (35)$$

be contained in $\text{Sep}^*(\mathcal{X} : \mathcal{Y})$ for $k = 1, 2, 3$.

Let $\Lambda_{k,\varepsilon} : L(\mathcal{Y}) \rightarrow L(\mathcal{X})$ be the unique linear map whose Choi representation satisfies $J(\Lambda_{k,\varepsilon}) = Q_{k,\varepsilon}$ for each $k = 1, 2, 3$. As discussed in Section 2, the block positivity of $Q_{k,\varepsilon}$ is equivalent to the positivity of $\Lambda_{k,\varepsilon}$. Consider first the case $k = 1$ and let

$$t = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}}. \quad (36)$$

A calculation reveals that

$$\Lambda_{1,\varepsilon}(Y) = \frac{\sqrt{1-\varepsilon^2}}{3} (\sigma_3 \otimes \mathbb{1}_{\mathcal{X}_2}) \Xi_t(Y) (\sigma_3 \otimes \mathbb{1}_{\mathcal{X}_2}), \quad (37)$$

where $\Xi_t : L(\mathcal{Y}) \rightarrow L(\mathcal{X})$ is the map defined in Lemma 3 and (in general)

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (38)$$

denote the Pauli operators. As $\varepsilon \in (0, 1)$, it holds that $t \in (0, \infty)$, and therefore Lemma 3 implies that $\Xi_t(Y) \in \text{Pos}(\mathcal{X})$ for every $Y \in \text{Pos}(Y)$. As we are simply conjugating $\Xi_t(Y)$ by a unitary and scaling it by a positive real factor, we also have that $\Lambda_{1,\varepsilon}(Y) \in \text{Pos}(\mathcal{X})$, for any $Y \in \text{Pos}(Y)$, which in turn implies that $Q_{1,\varepsilon} \in \text{Sep}^*(\mathcal{X} : \mathcal{Y})$.

For the case of $k = 2$ and $k = 3$, first define $U, V \in U(\mathbb{C}^2)$ as follows:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \quad (39)$$

These operators transform $\phi_1 = |\phi_1\rangle\langle\phi_1|$ into $\phi_2 = |\phi_2\rangle\langle\phi_2|$ and $\phi_3 = |\phi_3\rangle\langle\phi_3|$, respectively, and leave ϕ_4 unchanged, in the following sense:

$$\begin{aligned} (U^* \otimes U^*)\phi_1(U \otimes U) &= \phi_2, \\ (V^* \otimes V^*)\phi_1(V \otimes V) &= \phi_3, \\ (U^* \otimes U^*)\phi_4(U \otimes U) &= \phi_4, \\ (V^* \otimes V^*)\phi_4(V \otimes V) &= \phi_4. \end{aligned} \quad (40)$$

Therefore the following equations hold:

$$\begin{aligned} Q_{2,\varepsilon} &= (U^* \otimes \mathbb{1} \otimes U^* \otimes \mathbb{1}) Q_{1,\varepsilon} (U \otimes \mathbb{1} \otimes U \otimes \mathbb{1}), \\ Q_{3,\varepsilon} &= (V^* \otimes \mathbb{1} \otimes V^* \otimes \mathbb{1}) Q_{1,\varepsilon} (V \otimes \mathbb{1} \otimes V \otimes \mathbb{1}). \end{aligned} \quad (41)$$

It follows that $Q_{2,\varepsilon} \in \text{Sep}^*(\mathcal{X} : \mathcal{Y})$ and $Q_{3,\varepsilon} \in \text{Sep}^*(\mathcal{X} : \mathcal{Y})$, which completes the proof. \square

Remark 4. The upper bound obtained in Theorem 2 is achievable by an LOCC measurement, as it is the probability obtained by using the resource state $|\tau_\varepsilon\rangle$ to teleport the given Bell state from one player to the other, followed by an optimal local measurement to discriminate the resulting states.

Discriminating four Bell states

It is known that, for the perfect LOCC discrimination of all four Bell states using an auxiliary entangled state $|\tau_\varepsilon\rangle$ as above, one requires that $\varepsilon = 0$ (i.e., a maximally entangled pair of qubits is required). This fact follows from the method of [HSSH03], for instance. Here we prove a more precise bound on the optimal probability of a correct discrimination, for every choice of $\varepsilon \in [0, 1]$, along similar lines to the bound on three Bell states provided by Theorem 2. In the present case, in which all four Bell states are considered, the result is somewhat easier: one obtains an upper bound for PPT measurements that matches a bound that can be obtained by an LOCC measurement, implying that LOCC, separable, and PPT measurements are equivalent for this discrimination problem.

Theorem 5. *Let $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \mathbb{C}^2$, define $\mathcal{X} = \mathcal{X}_1 \otimes \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \otimes \mathcal{Y}_2$, and let $\varepsilon \in [0, 1]$. For any PPT measurement $\{P_1, P_2, P_3, P_4\} \subset \text{PPT}(\mathcal{X} : \mathcal{Y})$, the success probability of discriminating the states corresponding to the set*

$$\{|\phi_1\rangle \otimes |\tau_\varepsilon\rangle, |\phi_2\rangle \otimes |\tau_\varepsilon\rangle, |\phi_3\rangle \otimes |\tau_\varepsilon\rangle, |\phi_4\rangle \otimes |\tau_\varepsilon\rangle\} \subset (\mathcal{X}_1 \otimes \mathcal{Y}_1) \otimes (\mathcal{X}_2 \otimes \mathcal{Y}_2), \quad (42)$$

assuming a uniform distribution $p_1 = p_2 = p_3 = p_4 = 1/4$, is at most

$$\frac{1}{2} \left(1 + \sqrt{1 - \varepsilon^2}\right). \quad (43)$$

Proof. One may formulate a cone program corresponding to state discrimination by PPT measurements along similar lines to the cone program for separable measurements, simply by replacing the cone $\text{Sep}(\mathcal{X} : \mathcal{Y})$ by the cone $\text{PPT}(\mathcal{X} : \mathcal{Y})$ of positive semidefinite operators whose partial transpose is positive semidefinite. This cone program is a semidefinite program, as discussed in [Cos13], by virtue of the fact that partial transpose mapping is linear.

Consider the following operator:

$$H_\varepsilon = \frac{1}{8} \left[\mathbb{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} \otimes \tau_\varepsilon + \sqrt{1 - \varepsilon^2} \mathbb{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} \otimes T_{\mathcal{X}_2}(\phi_4) \right] \in \text{Herm}(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2). \quad (44)$$

It holds that

$$\text{Tr}(H_\varepsilon) = \frac{1}{2} \left(1 + \sqrt{1 - \varepsilon^2}\right), \quad (45)$$

so to complete the proof it suffices to prove that H_ε is dual feasible for the (semidefinite) cone program representing the PPT state discrimination problem under consideration. Dual feasibility will follow from the condition

$$(T_{\mathcal{X}_1} \otimes T_{\mathcal{X}_2}) \left(H_\varepsilon - \frac{1}{4} \phi_k \otimes \tau_\varepsilon \right) \in \text{Pos}(\mathcal{X}_1 \otimes \mathcal{Y}_1 \otimes \mathcal{X}_2 \otimes \mathcal{Y}_2) \quad (46)$$

(which is sufficient but not necessary for feasibility) for $k = 1, 2, 3, 4$. One may observe that

$$T_{\mathcal{X}_2}(\tau_\varepsilon) + \frac{\sqrt{1 - \varepsilon^2}}{2} \phi_4 = \frac{1}{2} \begin{pmatrix} 1 + \varepsilon & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1 - \varepsilon^2}}{2} & \frac{\sqrt{1 - \varepsilon^2}}{2} & 0 \\ 0 & \frac{\sqrt{1 - \varepsilon^2}}{2} & \frac{\sqrt{1 - \varepsilon^2}}{2} & 0 \\ 0 & 0 & 0 & 1 - \varepsilon \end{pmatrix} \quad (47)$$

is positive semidefinite, from which it follows that

$$(\mathbb{T}_{\mathcal{X}_1} \otimes \mathbb{T}_{\mathcal{X}_2}) \left(H_\varepsilon - \frac{1}{4} \phi_1 \otimes \tau_\varepsilon \right) = \frac{1}{4} \phi_4 \otimes \mathbb{T}_{\mathcal{X}_2}(\tau_\varepsilon) + \frac{\sqrt{1-\varepsilon^2}}{8} \mathbb{1}_{\mathcal{X}_1 \otimes \mathcal{Y}_1} \otimes \phi_4 \quad (48)$$

is also positive semidefinite. A similar calculation holds for $k = 2, 3, 4$, which completes the proof. \square

Remark 6. Similar to Theorem 2, one has that the upper bound obtained by Theorem 5 is optimal for LOCC measurements, as it is the probability obtained using teleportation.

4 State discrimination and unextendable product sets

In this section, we study the state discrimination problem for collections of states formed by unextendable product sets. An orthonormal collection of product vectors

$$\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y}, \quad (49)$$

for complex Euclidean spaces $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, is said to be an *unextendable product set* if it is impossible to find a nonzero product vector $u \otimes v \in \mathcal{X} \otimes \mathcal{Y}$ that is orthogonal to every element of \mathcal{A} [BDM⁺99]. That is, \mathcal{A} is an unextendable product set if, for every choice of vectors $u \in \mathcal{X}$ and $v \in \mathcal{Y}$ satisfying either $\langle u, u_k \rangle = 0$ or $\langle v, v_k \rangle = 0$ for each $k \in \{1, \dots, N\}$, one has that either $u = 0$ or $v = 0$ (or both).

Two subsections follow. The first subsection establishes a simple criterion for the states formed by any unextendable product set to be perfectly discriminated by separable measurements, and the second subsection proves that any set of states formed by taking the union of an unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ together with any pure state $z \in \mathcal{X} \otimes \mathcal{Y}$ orthogonal to every element of \mathcal{A} cannot be perfectly discriminated by a separable measurement. (It is evident that PPT measurements allow a perfect discrimination in both cases.)

A criterion for perfect separable discrimination of an unextendable product set

Here we provide a simple criterion for when an unextendable product set can be perfectly discriminated by separable measurements, and we use this criterion to show that there is an unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ that is not perfectly discriminated by any separable measurement when $\mathcal{X} = \mathcal{Y} = \mathbb{C}^4$. It is known that no unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ spanning a proper subspace of $\mathcal{X} \otimes \mathcal{Y}$ can be perfectly discriminated by an LOCC measurement [BDM⁺99], while every unextendable product set can be discriminated perfectly by a PPT measurement. It is also known that every unextendable product set $\mathcal{A} \subset \mathcal{X} \otimes \mathcal{Y}$ can be perfectly discriminated by separable measurements in the case $\mathcal{X} = \mathcal{Y} = \mathbb{C}^3$ [DMS⁺03].

The following notation will be used throughout this subsection. For $\mathcal{X} = \mathbb{C}^n$, $\mathcal{Y} = \mathbb{C}^m$, and $\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y}$ being an unextendable product set, we will write $\mathcal{A}_k = \mathcal{A} \setminus \{u_k \otimes v_k\}$, and define a set of rank-one product projections

$$\mathcal{P}_k = \{xx^* \otimes yy^* : x \in \mathcal{X}, y \in \mathcal{Y}, \|x\| = \|y\| = 1, \text{ and } x \otimes y \perp \mathcal{A}_k\} \quad (50)$$

for each $k = 1, \dots, N$. One may interpret each element $xx^* \otimes yy^*$ of \mathcal{P}_k as corresponding to a product vector $x \otimes y$ that could replace $u_k \otimes v_k$ in \mathcal{A} , yielding a (not necessarily unextendable) orthonormal product set.

The following theorem states that the sets $\mathcal{P}_1, \dots, \mathcal{P}_N$ defined above determine whether or not an unextendable product set can be perfectly discriminated by separable measurements.

Theorem 7. Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ be complex Euclidean spaces and let

$$\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y} \quad (51)$$

be an unextendable product set. The following two statements are equivalent:

1. There exists a separable measurement $\{P_1, \dots, P_N\} \subset \text{Sep}(\mathcal{X} : \mathcal{Y})$ that perfectly discriminates the states represented by \mathcal{A} (for any choice of nonzero probabilities p_1, \dots, p_N).
2. For $\mathcal{P}_1, \dots, \mathcal{P}_N$ as defined in (50), one has that the identity operator $\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}}$ can be written as a nonnegative linear combination of projections in the set $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_N$.

Proof. Assume first that statement 2 holds, so that one may write

$$\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} = \sum_{k=1}^N \sum_{j=1}^{M_k} \lambda_{k,j} x_{k,j} x_{k,j}^* \otimes y_{k,j} y_{k,j}^* \quad (52)$$

for some choice of positive integers M_1, \dots, M_N , nonnegative real numbers $\{\lambda_{k,j}\}$, and product vectors $\{x_{k,j} \otimes y_{k,j}\}$ satisfying

$$x_{k,j} x_{k,j}^* \otimes y_{k,j} y_{k,j}^* \in \mathcal{P}_k \quad (53)$$

for each $k \in \{1, \dots, N\}$ and $j \in \{1, \dots, M_k\}$. Define

$$P_k = \sum_{j=1}^{M_k} \lambda_{k,j} x_{k,j} x_{k,j}^* \otimes y_{k,j} y_{k,j}^* \quad (54)$$

for each $k \in \{1, \dots, N\}$. It is clear that $\{P_1, \dots, P_N\}$ is a separable measurement, and by the definition of the sets $\mathcal{P}_1, \dots, \mathcal{P}_N$ it necessarily holds that $\langle P_k, u_\ell u_\ell^* \otimes v_\ell v_\ell^* \rangle = 0$ when $k \neq \ell$. This implies that $\{P_1, \dots, P_N\}$ perfectly discriminates the states represented by \mathcal{A} , and therefore implies that statement 1 holds.

Now assume that statement 1 holds: there exists a separable measurement $\{P_1, \dots, P_N\}$ that perfectly discriminates the states represented by \mathcal{A} . As each measurement operator P_k is separable, it is possible to write

$$P_k = \sum_{j=1}^{M_k} \lambda_{k,j} x_{k,j} x_{k,j}^* \otimes y_{k,j} y_{k,j}^* \quad (55)$$

for some choice of nonnegative integers $\{M_k\}$, positive real numbers $\{\lambda_{k,j}\}$, and unit vectors $\{x_{k,j} : j = 1, \dots, M_k\} \subset \mathcal{X}$ and $\{y_{k,j} : j = 1, \dots, M_k\} \subset \mathcal{Y}$. The assumption that this measurement perfectly discriminates \mathcal{A} implies that $x_{k,j} \otimes y_{k,j} \perp \mathcal{A}_k$, and therefore $x_{k,j} x_{k,j}^* \otimes y_{k,j} y_{k,j}^* \in \mathcal{P}_k$, for each $k = 1, \dots, N$ and $j = 1, \dots, M_k$. As $P_1 + \dots + P_N = \mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}}$, it follows that statement 2 holds. \square

It is not immediately clear that Theorem 7 is useful for determining whether or not any particular unextendable product set can be discriminated by separable measurements, but indeed it is. What makes this so is the fact that each set \mathcal{P}_k is necessarily finite, as the following proposition establishes.

Proposition 8. Let \mathcal{X} and \mathcal{Y} be complex Euclidean spaces, let $\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y}$ be an unextendable product set, and let $\mathcal{P}_1, \dots, \mathcal{P}_N$ be as defined in (50). The sets $\mathcal{P}_1, \dots, \mathcal{P}_N$ are finite.

Proof. Assume toward contradiction that \mathcal{P}_k is infinite for some choice of $k \in \{1, \dots, N\}$. There are finitely many subsets $S \subseteq \{1, \dots, k-1, k+1, \dots, N\}$, so there must exist at least one such subset S with the property that there are infinitely many pairwise nonparallel product vectors of the form $x \otimes y$ such that $x \perp u_j$ for every $j \in S$ and $y \perp v_j$ for every $j \notin S$. This implies that both the subspace of \mathcal{X} orthogonal to $\{u_j : j \in S\}$ and the subspace of \mathcal{Y} orthogonal to $\{v_j : j \notin S\}$ have dimension at least 1, and at least one of them has dimension at least 2. It follows that there must exist a unit product vector $x \otimes y$ with three properties: (i) $x \perp u_j$ for every $j \in S$, (ii) $y \perp v_j$ for every $j \notin S$, and (iii) $x \otimes y \perp u_k \otimes v_k$. This contradicts the fact that \mathcal{A} is unextendable, and therefore completes the proof. \square

Given Proposition 8, it becomes straightforward to make use of Theorem 7 computationally. The sets $\mathcal{P}_1, \dots, \mathcal{P}_N$ can be computed by iterating over all $S \subseteq \{1, \dots, k-1, k+1, \dots, N\}$ and finding the (at most one) product state orthogonal to $\{u_j : j \in S\}$ on \mathcal{X} and $\{v_j : j \notin S\}$ on \mathcal{Y} . Then, the second statement in Theorem 7 can be checked through the use of linear programming (and even by hand in some cases).

Example

We now present an example of an unextendable product set in $\mathcal{X} \otimes \mathcal{Y}$, for $\mathcal{X} = \mathcal{Y} = \mathbb{C}^4$, that cannot be perfectly discriminated by separable measurements. In particular, let \mathcal{A} be the unextendable product set consisting of 8 states that were found in [Fen06]:

$$\begin{aligned}
|\phi_1\rangle &= |0\rangle|0\rangle, & |\phi_5\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|0\rangle - |1\rangle + |2\rangle)/3, \\
|\phi_2\rangle &= |1\rangle(|0\rangle - |2\rangle + |3\rangle)/\sqrt{3}, & |\phi_6\rangle &= (|0\rangle - |2\rangle + |3\rangle)|2\rangle/\sqrt{3}, \\
|\phi_3\rangle &= |2\rangle(|0\rangle + |1\rangle - |3\rangle)/\sqrt{3}, & |\phi_7\rangle &= (|0\rangle + |1\rangle - |3\rangle)|1\rangle/\sqrt{3}, \\
|\phi_4\rangle &= |3\rangle|3\rangle, & |\phi_8\rangle &= (|0\rangle - |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3.
\end{aligned} \tag{56}$$

For each $k = 1, \dots, 8$, there are exactly 6 product states contained in \mathcal{A}_k for each choice of k , which we represent by product vectors $|\phi_{k,j}\rangle$ for $j = 1, \dots, 6$. To be explicit, these states are as follows (where we have omitted normalization factors for brevity):

$$\begin{aligned}
|\phi_{1,1}\rangle &= |0\rangle|0\rangle, & |\phi_{1,4}\rangle &= (|0\rangle - |1\rangle + |2\rangle)(|0\rangle - |1\rangle + |2\rangle), \\
|\phi_{1,2}\rangle &= (|0\rangle + |1\rangle - |3\rangle)(|0\rangle + |2\rangle), & |\phi_{1,5}\rangle &= (|0\rangle + |2\rangle)(|0\rangle - |2\rangle + |3\rangle), \\
|\phi_{1,3}\rangle &= (|0\rangle - |1\rangle)(|0\rangle + |1\rangle - |3\rangle), & |\phi_{1,6}\rangle &= (|0\rangle - |2\rangle + |3\rangle)(|0\rangle - |1\rangle), \\
|\phi_{2,1}\rangle &= |1\rangle(|0\rangle - |2\rangle + |3\rangle), & |\phi_{2,4}\rangle &= (|0\rangle - |1\rangle + |2\rangle)(|1\rangle - 2|2\rangle + |3\rangle), \\
|\phi_{2,2}\rangle &= (|0\rangle + |1\rangle - |3\rangle)|2\rangle, & |\phi_{2,5}\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|0\rangle - |1\rangle - 2|2\rangle), \\
|\phi_{2,3}\rangle &= (|0\rangle + |1\rangle)|3\rangle, & |\phi_{2,6}\rangle &= (|1\rangle - |3\rangle)|0\rangle, \\
|\phi_{3,1}\rangle &= |2\rangle(|0\rangle + |1\rangle - |3\rangle), & |\phi_{3,4}\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|0\rangle + 2|1\rangle + |2\rangle), \\
|\phi_{3,2}\rangle &= (|0\rangle - |2\rangle + |3\rangle)|1\rangle, & |\phi_{3,5}\rangle &= (|0\rangle - |1\rangle + |2\rangle)(2|1\rangle - |2\rangle - |3\rangle), \\
|\phi_{3,3}\rangle &= (|2\rangle - |3\rangle)|0\rangle, & |\phi_{3,6}\rangle &= (|0\rangle - |2\rangle)|3\rangle, \\
|\phi_{4,1}\rangle &= |3\rangle|3\rangle, & |\phi_{4,4}\rangle &= (|2\rangle + |3\rangle)(|0\rangle - |2\rangle + |3\rangle), \\
|\phi_{4,2}\rangle &= (|0\rangle + |1\rangle - |2\rangle)(|2\rangle + |3\rangle), & |\phi_{4,5}\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|1\rangle + |2\rangle + |3\rangle), \\
|\phi_{4,3}\rangle &= (|1\rangle + |3\rangle)(|0\rangle + |1\rangle - |3\rangle), & |\phi_{4,6}\rangle &= (|0\rangle - |2\rangle + |3\rangle)(|1\rangle + |3\rangle), \\
|\phi_{5,1}\rangle &= (|1\rangle + |2\rangle + |3\rangle)(|0\rangle - |1\rangle + |2\rangle), & |\phi_{5,4}\rangle &= (|0\rangle - |2\rangle - 2|3\rangle)|2\rangle, \\
|\phi_{5,2}\rangle &= |1\rangle(2|0\rangle + |2\rangle - |3\rangle), & |\phi_{5,5}\rangle &= |2\rangle(2|0\rangle - |1\rangle + |3\rangle), \\
|\phi_{5,3}\rangle &= |3\rangle|0\rangle, & |\phi_{5,6}\rangle &= (|0\rangle + |1\rangle + 2|3\rangle)|1\rangle,
\end{aligned}$$

$$\begin{aligned}
|\phi_{6,1}\rangle &= (|0\rangle - |2\rangle + |3\rangle) |2\rangle, & |\phi_{6,4}\rangle &= (|0\rangle - |1\rangle - 2|2\rangle) (|1\rangle + |2\rangle + |3\rangle), \\
|\phi_{6,2}\rangle &= |3\rangle (|0\rangle - |2\rangle), & |\phi_{6,5}\rangle &= |2\rangle (|0\rangle - |2\rangle + |3\rangle), \\
|\phi_{6,3}\rangle &= |0\rangle (|2\rangle - |3\rangle), & |\phi_{6,6}\rangle &= (|1\rangle - 2|2\rangle + |3\rangle) (|0\rangle - |1\rangle + |2\rangle), \\
|\phi_{7,1}\rangle &= (|0\rangle + |1\rangle - |3\rangle) |1\rangle, & |\phi_{7,4}\rangle &= (|0\rangle + 2|1\rangle + |2\rangle) (|1\rangle + |2\rangle + |3\rangle), \\
|\phi_{7,2}\rangle &= |0\rangle (|1\rangle - |3\rangle), & |\phi_{7,5}\rangle &= (2|1\rangle - |2\rangle - |3\rangle) (|0\rangle - |1\rangle + |2\rangle), \\
|\phi_{7,3}\rangle &= |1\rangle (|0\rangle + |1\rangle - |3\rangle), & |\phi_{7,6}\rangle &= |3\rangle (|0\rangle + |1\rangle), \\
|\phi_{8,1}\rangle &= (|0\rangle - |1\rangle + |2\rangle) (|0\rangle + |1\rangle + |2\rangle), & |\phi_{8,4}\rangle &= |0\rangle |3\rangle, \\
|\phi_{8,2}\rangle &= |1\rangle (|0\rangle - |2\rangle - 2|3\rangle), & |\phi_{8,5}\rangle &= (2|0\rangle + |2\rangle - |3\rangle) |2\rangle, \\
|\phi_{8,3}\rangle &= (2|0\rangle - |1\rangle + |3\rangle) |1\rangle, & |\phi_{8,6}\rangle &= |2\rangle (|0\rangle + |1\rangle + 2|3\rangle).
\end{aligned}$$

One may verify by a computer that $\mathbb{1} \otimes \mathbb{1}$ is not contained in the convex cone generated by

$$\{|\phi_{k,j}\rangle\langle\phi_{k,j}| : k = 1, \dots, 8, j = 1, \dots, 6\}. \quad (57)$$

(In fact, $\mathbb{1} \otimes \mathbb{1}$ is not in the linear span of the set (57).) Theorem 7 therefore implies that this unextendable product set is not perfectly discriminated by separable measurements.

Impossibility to discriminate an unextendable product set plus one more pure state

Next, we prove an upper bound on the probability to correctly discriminate any unextendable product set, together with one extra pure state orthogonal to the members of the unextendable product set, by a separable measurement. Central to the proof of this statement is a family of positive linear maps previously studied in the literature [Ter01, BGR05].

Before proving this fact, we note that it is fairly straightforward to obtain a qualitative result along similar lines: if a separable measurement were able to perfectly discriminate a particular product set from any state orthogonal to this product set, there would necessarily be a separable measurement operator orthogonal to the space spanned by the product set, implying that some nonzero product state must be orthogonal to the product set (and therefore the product set must be extendable). Related results based on this sort of argument may be found in [Ban11]. An advantage of the method described in the present paper is that one obtains precise bounds on the optimal discrimination probability, as opposed to a statement that a perfect discrimination is not possible.

The following lemma is required for the proof of the theorem below.

Lemma 9 (Terhal). *For given complex Euclidean spaces $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$, and any unextendable product set*

$$\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y}, \quad (58)$$

there exists a positive real number $\lambda_{\mathcal{A}} > 0$ such that

$$(\mathbb{1}_{\mathcal{X}} \otimes y^*) \left(\sum_{k=1}^N u_k u_k^* \otimes v_k v_k^* \right) (\mathbb{1}_{\mathcal{X}} \otimes y) - \lambda_{\mathcal{A}} \|y\|^2 \mathbb{1}_{\mathcal{X}} \in \text{Pos}(\mathcal{X}), \quad (59)$$

for every $y \in \mathcal{Y}$.

A proof of the lemma, as well as a constructive procedure to calculate a bound on $\lambda_{\mathcal{A}}$, can be found in [Ter01].

Theorem 10. Let $\mathcal{X} = \mathbb{C}^n$ and $\mathcal{Y} = \mathbb{C}^m$ be complex Euclidean spaces, let

$$\mathcal{A} = \{u_k \otimes v_k : k = 1, \dots, N\} \subset \mathcal{X} \otimes \mathcal{Y} \quad (60)$$

be an unextendable product set, and let $z \in \mathcal{X} \otimes \mathcal{Y}$ be a unit vector orthogonal to \mathcal{A} . Assuming a uniform selection, the probability to correctly discriminate the states corresponding to the set $\mathcal{A} \cup \{z\}$ by a separable measurement, assuming a uniform selection of states, is upper-bounded by

$$1 - \frac{\lambda_{\mathcal{A}}}{(N+1)\delta'}, \quad (61)$$

where $\lambda_{\mathcal{A}}$ is a positive real number satisfying the requirements of Lemma 9 and $\delta = \|\text{Tr}_{\mathcal{X}}(zz^*)\|$.

Proof. Consider the following Hermitian operator:

$$H = \frac{1}{N+1} \left(\sum_{k=1}^N u_k u_k^* \otimes v_k v_k^* + \left(1 - \frac{\lambda_{\mathcal{A}}}{\delta}\right) zz^* \right). \quad (62)$$

We want to show that H is a feasible solution of the dual problem (15) for the state discrimination problem under consideration. It is clear that

$$H - \frac{1}{N+1} u_k u_k^* \otimes v_k v_k^* \in \text{Pos}(\mathcal{X} \otimes \mathcal{Y}) \subset \text{Sep}^*(\mathcal{X} : \mathcal{Y}) \quad (63)$$

for $k = 1, \dots, N$. The remaining constraint left to be checked is the following:

$$H - \frac{1}{N+1} zz^* = \frac{1}{N+1} \left(\sum_{k=1}^N u_k u_k^* \otimes v_k v_k^* - \frac{\lambda_{\mathcal{A}}}{\delta} zz^* \right) \in \text{Sep}^*(\mathcal{X} : \mathcal{Y}). \quad (64)$$

Using the fact that

$$\delta \|y\|^2 \mathbb{1}_{\mathcal{X}} - (\mathbb{1}_{\mathcal{X}} \otimes y^*) zz^* (\mathbb{1}_{\mathcal{X}} \otimes y) \in \text{Pos}(\mathcal{X}), \quad (65)$$

for any $y \in \mathcal{Y}$, together with Lemma 9, one has that

$$(\mathbb{1} \otimes y)^* \left(\sum_{k=1}^N u_k u_k^* \otimes v_k v_k^* - \frac{\lambda_{\mathcal{A}}}{\delta} zz^* \right) (\mathbb{1} \otimes y) \in \text{Pos}(\mathcal{X}) \quad (66)$$

and therefore the constraint (65) is satisfied. Finally, it holds that

$$\text{Tr}(H) = 1 - \frac{\lambda_{\mathcal{A}}}{(N+1)\delta'}, \quad (67)$$

which completes the proof. \square

Example

Theorem 10 allow us to find specific bounds for the probability of correctly discriminating certain sets of states. For instance, here we consider the following unextendable product set in $\mathcal{X} \otimes \mathcal{Y}$ for $\mathcal{X} = \mathcal{Y} = \mathbb{C}^3$, commonly known as the *tiles set*:

$$\begin{aligned} |\phi_1\rangle &= |0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right), & |\phi_2\rangle &= |2\rangle \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}} \right), & |\phi_3\rangle &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) |2\rangle, & |\phi_4\rangle &= \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}} \right) |0\rangle, \\ |\phi_5\rangle &= \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle) (|0\rangle + |1\rangle + |2\rangle). \end{aligned} \quad (68)$$

For a pure state orthogonal to this set, one may take

$$|\psi\rangle = \frac{1}{2} (|0\rangle|0\rangle + |0\rangle|1\rangle - |0\rangle|2\rangle - |1\rangle|2\rangle). \quad (69)$$

Using the procedure described in [Ter01], one obtains

$$\lambda_{\mathcal{A}} \geq \frac{1}{9} \left(1 - \sqrt{\frac{5}{6}}\right)^2. \quad (70)$$

Therefore, if we assume that each state is selected with probability $1/6$, the maximum probability of correctly discriminating the set $\{|\phi_1\rangle, \dots, |\phi_5\rangle, |\psi\rangle\}$ by a separable measurement is at most

$$1 - \frac{1}{54} \frac{\left(1 - \sqrt{\frac{5}{6}}\right)^2}{\cos\left(\frac{\pi}{8}\right)^2} < 1 - 1.647 \times 10^{-4}. \quad (71)$$

5 An optimal bound on discriminating the Yu–Duan–Ying states

In this section we prove a tight bound of $3/4$ on the maximum success probability for any LOCC measurement to discriminate the set of states (6) exhibited by Yu, Duan, and Ying [YDY12], assuming a uniform selection of states. The fact that this bound can be achieved by an LOCC measurement is trivial: if Alice and Bob measure their parts of the states with respect to the standard basis, they can easily discriminate $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_4\rangle$, erring only in the case that they receive $|\phi_3\rangle$. The fact that this bound is optimal will be proved by exhibiting a feasible solution H to the dual problem (15), corresponding to the state discrimination problem at hand, such that $\text{Tr}(H) = 3/4$.

It is convenient for the analysis that follows to make use of the correspondence between operators and vectors given by the linear function defined by the action

$$\text{vec}(|k\rangle\langle j|) = |k\rangle|j\rangle \quad (72)$$

on standard basis vectors. With respect to this correspondence, the states (6) are given by tensor products of the Pauli operators (38) as follows:

$$|\phi_1\rangle = \frac{1}{2} \text{vec}(U_1), \quad |\phi_2\rangle = \frac{1}{2} \text{vec}(U_2), \quad |\phi_3\rangle = \frac{1}{2} \text{vec}(U_3), \quad \text{and} \quad |\phi_4\rangle = \frac{1}{2} \text{vec}(U_4), \quad (73)$$

for

$$U_1 = \sigma_0 \otimes \sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \sigma_1 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (74)$$

$$U_3 = i\sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad U_4 = \sigma_3 \otimes \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (75)$$

A feasible solution of the dual problem (15) is based on a construction of block positive operators that correspond, via the Choi isomorphism, to the family of positive maps introduced by Breuer and Hall [Bre06, Hal06].

Proposition 11 (Breuer–Hall). *Let $\mathcal{X} = \mathcal{Y} = \mathbb{C}^n$ and let $U, V \in \mathcal{U}(\mathcal{Y}, \mathcal{X})$ be unitary operators such that $U^\top V \in \mathcal{U}(\mathcal{Y})$ is skew-symmetric: $(V^\top U)^\top = -V^\top U$. It holds that*

$$\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - \text{vec}(U) \text{vec}(U)^* - \text{T}_{\mathcal{X}}(\text{vec}(V) \text{vec}(V)^*) \in \text{Sep}^*(\mathcal{X} : \mathcal{Y}). \quad (76)$$

Proof. For every unit vector $y \in \mathcal{Y}$, one has

$$\begin{aligned} (\mathbb{1}_{\mathcal{X}} \otimes y^*)(\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - \text{vec}(U) \text{vec}(U)^* - \text{T}_{\mathcal{X}}(\text{vec}(V) \text{vec}(V)^*))(\mathbb{1}_{\mathcal{X}} \otimes y) \\ = \mathbb{1}_{\mathcal{X}} - U\bar{y}y^\top U^* - \bar{V}y y^* V^\top. \end{aligned} \quad (77)$$

As it holds that $V^\top U$ is skew-symmetric, we have

$$\langle \bar{V}y, U\bar{y} \rangle = y^* V^\top U \bar{y} = \langle y y^\top, V^\top U \rangle = 0, \quad (78)$$

as the last inner product is between a symmetric and a skew-symmetric operator. Because U and V are unitary, it follows that $U\bar{y}y^\top U^* + \bar{V}y y^* V^\top$ is a rank two orthogonal projection, so the operator represented by (77) is also a projection and is therefore positive semidefinite. \square

Remark 12. The assumption of Proposition 11 requires n to be even, as skew-symmetric unitary operators exist only in even dimensions.

Now, for one of the four states $\rho_1 = |\phi_1\rangle\langle\phi_1|$, $\rho_2 = |\phi_2\rangle\langle\phi_2|$, $\rho_3 = |\phi_3\rangle\langle\phi_3|$, or $\rho_4 = |\phi_4\rangle\langle\phi_4|$ drawn with uniform probabilities $p_1 = \dots = p_4 = 1/4$, one has that the following operator is a feasible solution to the dual problem (15):

$$H = \frac{1}{16}(\mathbb{1}_{\mathcal{X}} \otimes \mathbb{1}_{\mathcal{Y}} - \text{T}_{\mathcal{X}}(\text{vec}(V) \text{vec}(V)^*)) \quad (79)$$

for

$$V = i\sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (80)$$

Due to Proposition 11, the feasibility of H follows from the condition

$$(V^\top U_k)^\top = -V^\top U_k, \quad (81)$$

which can be checked by inspecting each of the four cases. It is easy to calculate that $\text{Tr}(H) = 3/4$, and so the required bound has been obtained.

6 Conclusion

In this paper we have used techniques from convex optimization, and cone programming in particular, to study the limitations of separable measurements for the task of discriminating sets of bipartite state.

Several interesting questions regarding the discrimination of sets of bipartite states by means of separable and LOCC measurements remain unsolved. Among them are the following two questions.

- In Section 3 we proved a tight bound on the entanglement cost of discriminating sets of Bell states by means of LOCC protocols. More generally, one could ask how much entanglement it costs to distinguish maximally entangled states in $\mathbb{C}^n \otimes \mathbb{C}^n$.
- Ghosh et al. [GKRS04] have shown that orthogonal maximally entangled states, which are in canonical form, can always be discriminated, by means of LOCC protocols, if two copies of each of the states are provided. The question of whether two copies are sufficient to discriminate any set of orthogonal pure states is open even for separable and PPT measurements.

The techniques presented in the paper are not intrinsically limited to the setting of bipartite pure states—applications of these techniques to mixed states and multipartite states are topics for possible future work.

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