Hadamard-Diagonalizable Graphs with Perfect Quantum State Transfer

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Mount Allison University

Workshop on Algebraic Graph Theory & Quantum Walks
Waterloo, Ontario, Canada

April 25, 2018

Hadamard matrices

- A **Hadamard Matrix** H is an $n \times n$ matrix whose entries are all 1 or -1 and satisfies $HH^T = nI$ (equivalently, its rows and/or columns are mutually orthogonal).
- The *standard* Hadamard matrices of order 2^n are

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H_4 = H_2 \otimes H_2 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

$$H_{2^{n+1}} = H_2 \otimes H_{2^n} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix} \quad \text{for all} \quad n \geq 1.$$

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Adjacency matrix of a (weighted) graph

- The **adjacency matrix** of a (weighted) graph is the $n \times n$ matrix $A = (a_{j,k})$ defined by

$$a_{j,k} = \begin{cases} w(j,k) & \text{if j and k are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

- For example, the graph on the left below has adjacency matrix on the right:

$$A = \begin{bmatrix} 0 & 5 & 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 4 & 0 & 0 \\ 3 & 0 & 0 & 1 & 2 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 & 0 \end{bmatrix}$$

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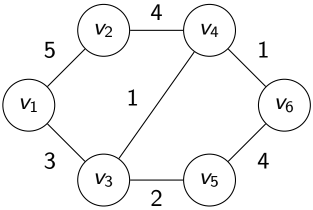

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Laplacian matrix of a (weighted) graph

- The **Laplacian matrix** of a (weighted) graph is the $n \times n$ matrix $L = D - A$ where A is its adjacency matrix and D is its diagonal degree matrix.

$$D = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix},$$

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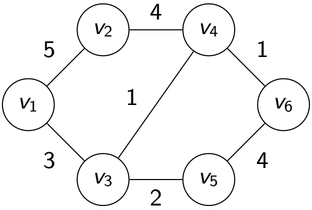

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The Laplacian matrix L of any graph has many nice properties:

- Symmetric.
- Positive semidefinite (since it is diagonally dominant).
- Row sums 0. Equivalently...
- $(1, 1, \ldots, 1)$ is an eigenvector with eigenvalue 0.
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Perfect State Transfer

A graph with Laplacian L exhibits **perfect state transfer (PST)** at time t between vertices v_j and v_k if the (j,k)-entry of e^{itL} has magnitude 1.

- Since L is symmetric, e^{itL} is unitary, so none of its entries have magnitude larger than 1. Also, if PST occurs between vertices v_j and v_k then all other entries in the j-th row and k-th column of e^{itL} are 0.
- Motivation: This means that after time t, the quantum state $|j\rangle$ evolves into the state $e^{itL}|j\rangle = |k\rangle$.
- Morally, this means we have transferred the quantum state from vertex v_j to vertex v_k of the graph (perfectly—without any noise/errors).

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Quantum State Transfer

Question (The BIG Question)

Which (weighted) graphs exhibit PST?

- P_2 and P_3.
- Square, cube, hypercube.
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A graph with Laplacian L is **Hadamard-diagonalizable** if there exists a Hadamard matrix H such that $H^T LH$ is diagonal.

- It is more convenient to diagonalize by a scaled Hadamard $U = \frac{1}{\sqrt{n}} H$ so that $U^T LU$ contains the eigenvalues of L along its diagonal.
- For example, the square graph is Hadamard-diagonalizable:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} = \left(\frac{1}{2} H_4\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \left(\frac{1}{2} H_4\right)^T.$$

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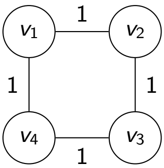

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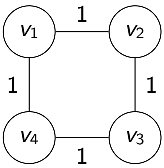

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Hadamard-Diagonalizable Graphs

WLOG, H can be assumed to have every entry in its first row and column equal to 1.

- Barik–Fallat–Kirkland (2011) says if a graph G has integer weights and is Hadamard-diagonalizable then:
  - G is regular (i.e., the sum of the edge weights adjacent to each vertex is constant).
  - The eigenvalues of its Laplacian are even integers.
- Integer weights can be extended to rational weights via scaling (changes the time of PST).

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Relationship with Cubelike Graphs

Lemma (Hadamard-Diagonalizability and Cubelike Graphs)

An unweighted graph is diagonalizable by the standard Hadamard if and only if it is cubelike.

- Hadamard diagonalizability is thus more general than being cubelike.
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A Spectral Characterisation

Given a spectral decomposition of a Hadamard-diagonalizable graph, the following theorem shows that it is easy to determine whether or not PST occurs at time $\pi/2$.

Theorem (PST from Spectrum)

Let G be an integer-weighted graph that is diagonalizable by a Hadamard matrix $H = (h_{i,j})$. Denote the eigenvalues of its Laplacian by $\lambda_1, \cdots, \lambda_n$. Then G has PST between vertices v_j and v_k at time $\pi/2$ if and only if, for each $\ell = 1, \cdots, n$,

$$\lambda_\ell \equiv 1 - h_{j,\ell} h_{k,\ell} \ (mod\ 4).$$

Corollary: If PST occurs, then half of the eigenvalues are 0 (mod 4) and the other half are 2 (mod 4).

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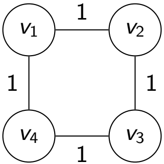

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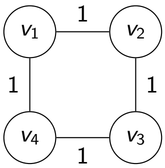

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This result can help us create graphs with PST.

- E.g., There is no unweighted Hadamard-diagonalizable graph on 12 vertices that has PST. But...
- The (essentially unique) 12×12 Hadamard is

$$H_{12} = \begin{bmatrix}
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1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1
\end{bmatrix}$$

A Spectral Characterisation

To make a graph with PST between vertices v_1 and v_2, we construct a set of integer eigenvalues that are 0 or 2 (mod 4) according to the second row of H_{12}:

$$1, -1, 1, -1, 1, 1, 1, -1, -1, -1, 1, -1.$$

One possible choice:

$$0, 18, 24, 18, 24, 24, 24, 18, 18, 18, 12, 18.$$

Then we set $\Lambda = \mathrm{diag}(0, 18, 24, 18, 24, 24, 24, 18, 18, 18, 12, 18)$ and

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It is straightforward to calculate

$$
\begin{aligned}
L &= \frac{1}{12} H_{12}^{T} \Lambda H_{12} \\
&= \begin{bmatrix}
18 & 0 & -1 & -1 & -1 & -3 & -3 & -3 & -1 & -3 & -1 & -1 \\
0 & 18 & -1 & -1 & -1 & -3 & -3 & -3 & -1 & -3 & -1 & -1 \\
-1 & -1 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 & -4 & -2 \\
-1 & -1 & -2 & 18 & -4 & 0 & 0 & -2 & -2 & -2 & -2 & -2 \\
-1 & -1 & -2 & -4 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 \\
-3 & -3 & 0 & 0 & -2 & 18 & -2 & -2 & 0 & -2 & -2 & -2 \\
-3 & -3 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 & 0 \\
-3 & -3 & 0 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 \\
-1 & -1 & -2 & -2 & -2 & 0 & -2 & -2 & 18 & 0 & -2 & -4 \\
-3 & -3 & -2 & -2 & 0 & -2 & -2 & -2 & 0 & 18 & 0 & -2 \\
-1 & -1 & -4 & -2 & -2 & -2 & 0 & -2 & -2 & 0 & 18 & -2 \\
-1 & -1 & -2 & -2 & -2 & -2 & 0 & 0 & -4 & -2 & -2 & 18
\end{bmatrix}
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This is (necessarily, by construction) the Laplacian of a graph with PST.

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$$= \begin{bmatrix} 18 & 0 & -1 & -1 & -1 & -3 & -3 & -3 & -1 & -3 & -1 & -1 \\ 0 & 18 & -1 & -1 & -1 & -3 & -3 & -3 & -1 & -3 & -1 & -1 \\ -1 & -1 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 & -4 & -2 \\ -1 & -1 & -2 & 18 & -4 & 0 & 0 & -2 & -2 & -2 & -2 & -2 \\ -1 & -1 & -2 & -4 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 \\ -3 & -3 & 0 & 0 & -2 & 18 & -2 & -2 & 0 & -2 & -2 & -2 \\ -3 & -3 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 & 0 \\ -3 & -3 & 0 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 \\ -1 & -1 & -2 & -2 & -2 & 0 & -2 & -2 & 18 & 0 & -2 & -4 \\ -3 & -3 & -2 & -2 & 0 & -2 & -2 & -2 & 0 & 18 & 0 & -2 \\ -1 & -1 & -4 & -2 & -2 & -2 & 0 & -2 & -2 & 0 & 18 & -2 \\ -1 & -1 & -2 & -2 & -2 & -2 & 0 & 0 & -4 & -2 & -2 & 18 \end{bmatrix}$$

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-1 & -1 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 & -4 & -2 \\
-1 & -1 & -2 & 18 & -4 & 0 & 0 & -2 & -2 & -2 & -2 & -2 \\
-1 & -1 & -2 & -4 & 18 & -2 & -2 & 0 & -2 & 0 & -2 & -2 \\
-3 & -3 & 0 & 0 & -2 & 18 & -2 & -2 & 0 & -2 & -2 & -2 \\
-3 & -3 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 & 0 \\
-3 & -3 & 0 & -2 & 0 & -2 & -2 & 18 & -2 & -2 & -2 & 0 \\
-1 & -1 & -2 & -2 & -2 & 0 & -2 & -2 & 18 & 0 & -2 & -4 \\
-3 & -3 & -2 & -2 & 0 & -2 & -2 & -2 & 0 & 18 & 0 & -2 \\
-1 & -1 & -4 & -2 & -2 & -2 & 0 & -2 & -2 & 0 & 18 & -2 \\
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This is (necessarily, by construction) the Laplacian of a graph with PST.

Merge of Graphs

Suppose that G_1 and G_2 are two weighted graphs that are both diagonalizable by a Hadamard matrix H, with Laplacians $L_1 = D_1 - A_1$ and $L_2 = D_2 - A_2$, respectively.

We define their **merge** with respect to the weights w_1 and w_2 to be the graph $G_1 \,{}_{w_1}\!\odot_{w_2} G_2$ with Laplacian

$$\begin{bmatrix} w_1 L_1 + w_2 D_2 & -w_2 A_2 \\ -w_2 A_2 & w_1 L_1 + w_2 D_2 \end{bmatrix}.$$

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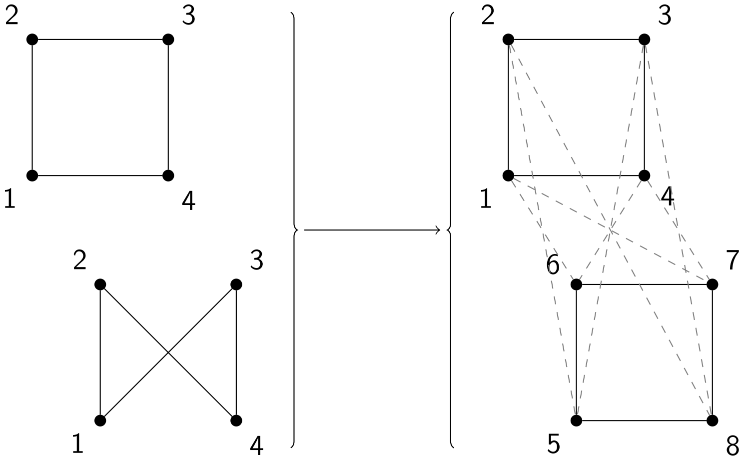
Example



Using the Merge to Create Graphs with PST

Our main interest in the merge comes from the fact that we can use it to create larger graphs with PST from smaller ones.

Theorem (Merge for PST)

Suppose G_1 and G_2 are integer-weighted graphs with regularities d_1 and d_2, respectively, both of which are diagonalizable by the same Hadamard matrix H. Then $G_1 \,{}_{w_1}\!\odot_{w_2}\, G_2$ has PST between vertices v_j and v_k at time $\pi/2$ if and only if one of the following 8 conditions holds:

1) $j, k \in \{1, \ldots, n\}$ and
   - 1.a) w_1 is odd, w_2 is even, and G_1 has PST between v_j and v_k at time $\pi/2$, or
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Corollary (PST of Weighted Hypercube)

Suppose $w_1, w_2, \ldots, w_n$ are nonzero integers, exactly d of which are odd, and consider the weighted hypercube $C_n := (w_1 K_2)\square(w_2 K_2)\square\cdots\square(w_n K_2)$. For each vertex v_j of C_n, there is a vertex v_k at distance d from v_j such that there is PST from v_j to v_k at time $\pi/2$.

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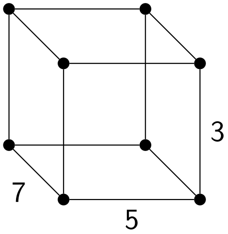

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- Are there Hadamard-diagonalizable graphs with PST of all sizes that are a multiple of 4?
- Is there a Hadamard-diagonalizable graph with PST associated with each Hadamard matrix?
- If G is unweighted, Hadamard diagonalizable, r-regular, has n vertices, and has PST at time $\pi/2$. Is it the case that $n \leq 2^r$?
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