# The Spectra Arising from Positive Linear Maps

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# Positive Maps

**Definition**

*A linear map $\Phi : M_n \to M_n$ is called **positive** if $\Phi(X)$ is (Hermitian) positive semidefinite (PSD) whenever $X$ is.*

- For example, the transpose map is positive.
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# Positive Maps

A positive map $\Phi$ is not necessarily **completely positive**. That is, there might exist PSD $X \in M_m \otimes M_n$ such that $(I_m \otimes \Phi)(X)$ is not PSD.

- The "standard example" is the transpose map $T : M_2 \to M_2$:

$$(I_2 \otimes T)\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- The above matrix has eigenvalues $1, 1, 1$, and $-1$, so it is not positive semidefinite.

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# Simple Spectral Inequalities

**Question**

*What are the possible spectra of matrices of the form $(I \otimes \Phi)(X)$ when $\Phi$ is positive and $X$ is PSD?*

Equivalently, what are the possible spectra of **entanglement witnesses**?

That is, Hermitian matrices $W \in M_m \otimes M_n$ such that

$$(\langle a| \otimes \langle b|)W(|a\rangle \otimes |b\rangle) \geq 0 \quad \text{for all} \quad |a\rangle \in \mathbb{C}^m, |b\rangle \in \mathbb{C}^n.$$

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# Simple Spectral Inequalities

For example, can entanglement witnesses $W \in M_2 \otimes M_2$ have more than one negative eigenvalue?

**Theorem**

*If $W \in M_m \otimes M_n$ is an entanglement witness, then it has no more than $(m-1)(n-1)$ negative eigenvalues.*

- Follows from the fact that entangled subspaces can have dimension no larger than $(m-1)(n-1)$.
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**Theorem (J.–Kribs, 2010)**

*If $W \in M_m \otimes M_n$ is an entanglement witness, then*

$$\lambda_{\min}(W)/\lambda_{\max}(W) \geq 1 - \min\{m, n\}.$$

- Proof is straightforward.
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# Two-Qubit Entanglement Witnesses

Can we do better? Well, in small dimensions...

**Theorem (J.–Patterson)**

*There exists an entanglement witness in $M_2 \otimes M_2$ with eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4$ if and only if the following inequalities hold:*

- $\mu_3 \geq 0$,
- $\mu_4 \geq -\mu_2$, *and*
- $\mu_4 \geq -\sqrt{\mu_1 \mu_3}$.

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We can visualize the set of possible spectra by scaling $W$ so that $\mathrm{Tr}(W) = 1$. Then $\mu_4 = 1 - \mu_1 - \mu_2 - \mu_3$ and the (unsorted) $(\mu_1, \mu_2, \mu_3)$ region looks like:

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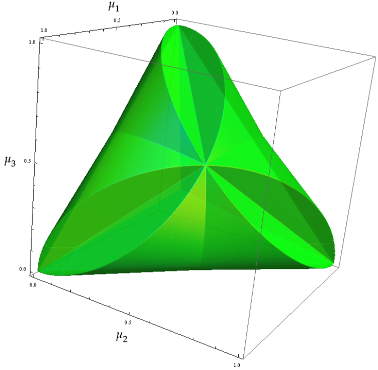

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**Proof sketch:**

- Every entanglement witness $W \in M_2 \otimes M_2$ can be written in the form $W = X + (I \otimes T)(Y)$, where $X, Y \in M_2 \otimes M_2$ are PSD
- If $Y = |v\rangle\langle v|$ is PSD with rank 1, eigenvalues of $(I \otimes T)(Y)$ are easy to compute in terms of the Schmidt coefficients of $|v\rangle$.
- Eigenvalues of $W$ are no smaller than those of $(I \otimes T)(Y)$. Done.

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# Qubit–Qudit Entanglement Witnesses

Next, we consider entanglement witnesses $W \in M_2 \otimes M_n$, where $n \geq 2$.

- This problem is much harder. Even when $n = 3$, a complete characterization is beyond us.
- To simplify things, we instead characterize the possible convex combinations of (unsorted) spectra of entanglement witnesses (we denote this set by $Conv\left(\sigma(\mathrm{EW}_{m,n})\right)$).
- For example, $(4, 2, 1, -2) \in \sigma(\mathrm{EW}_{2,2})$, so

$$(4, 2, 1, -2) + (4, 2, -2, 1) = (8, 4, -1, -1) \in Conv\left(\sigma(\mathrm{EW}_{2,2})\right)$$
$$\notin \sigma(\mathrm{EW}_{2,2}).$$

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**Theorem (J.–Patterson)**

*Suppose $\vec{\mu} \in \mathbb{R}^{2n}$. Define $s_k := \sum_{j=k}^{2n} \mu_j^{\downarrow}$ for $k = 1, 2, 3$ and $s_- := \sum_{\{j:\mu_j<0\}} \mu_j$. Then the following are equivalent:*

(a) $\vec{\mu} \in Conv\big(\sigma(\mathrm{EW}_{2,n})\big)$.

(b) *There exists a real PSD matrix $X \in M_2$ such that*

$$x_{1,1} + x_{2,2} \leq s_1, \quad x_{2,2} \leq s_2, \quad x_{1,2} + x_{2,2} \leq s_3, \quad \textit{and} \quad x_{1,2} \leq s_-.$$

(c) *If we define $q_1 := s_1^2 - 4s_-^2$ and $q_2 := (s_1 + 2s_3)^2 - 8s_3^2$ then:*

$$q_1, q_2 \geq 0$$
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# Qubit–Qudit Entanglement Witnesses

Each of the inequalities described by part (c) of that theorem is a necessary condition that the spectra of entanglement witnesses must satisfy.

- These inequalities are not sufficient, even if $n = 2$.
- However, they are considerably stronger than all previously-known necessary conditions.
- Exact necessary and sufficient conditions are likely unreasonable to hope for (even inverse eigenvalue problems for "simple" matrices like entrywise non-negative matrices are very hard).

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- These inequalities are not sufficient, even if $n = 2$.
- However, they are considerably stronger than all previously-known necessary conditions.
- Exact necessary and sufficient conditions are likely unreasonable to hope for (even inverse eigenvalue problems for "simple" matrices like entrywise non-negative matrices are very hard).

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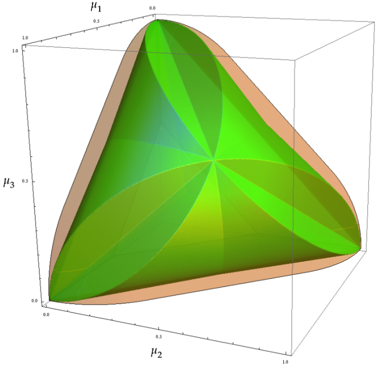

# Decomposable Entanglement Witnesses

When going to even higher dimensions ($M_m \otimes M_n$ with $m, n \geq 3$), we have to sacrifice even more.

- Our methods now only work for decomposable entanglement witnesses: those of the form $W = X + (I \otimes T)(Y)$, with $X$ and $Y$ positive semidefinite.
- Equivalently, positive maps of the form $\Phi = \Psi_1 + T \circ \Psi_2$, where $\Psi_1, \Psi_2$ are completely positive.
- We can characterize the set $Conv\left(\sigma(\mathrm{DEW}_{m,n})\right)$ (DEW stands for "decomposable entanglement witness") for all $m, n$ (but the theorem is too ugly for a 25-minute talk).

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# Decomposable Entanglement Witnesses

For example, $\vec{\mu} \in Conv\left(\sigma(\mathrm{DEW}_{3,3})\right)$ if and only if there exist real PSD matrices $X, Y \in M_3$ such that...

$$\begin{aligned}
(x_{1,1} + x_{2,2} + x_{3,3}) + (y_{1,1} + y_{2,2} + y_{3,3}) &\leq s_1 \\
(x_{2,2} + x_{3,3}) + (y_{2,2} + y_{3,3}) &\leq s_2 \\
(x_{2,2} + x_{3,3} - x_{1,2}) + (y_{2,2} + y_{3,3} - y_{1,2}) &\leq s_3 \\
(x_{3,3} - x_{1,2}) + (y_{2,2} + y_{3,3} - y_{1,2} - y_{1,3}) &\leq s_4 \\
(x_{3,3} - x_{1,2} - x_{1,3}) + (y_{3,3} - y_{1,2} - y_{1,3}) &\leq s_5 \\
(x_{3,3} - x_{1,2} - x_{1,3} - x_{2,3}) + (y_{3,3} - y_{1,2} - y_{1,3} - y_{2,3}) &\leq s_6 \\
(-x_{1,2} - x_{1,3} - x_{2,3}) + (-y_{1,2} - y_{1,3} - y_{2,3}) &\leq s_7 \\
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# Entanglement Witnesses in Higher Dimensions

We know comparatively little about (non-decomposable) entanglement witnesses when $m, n \geq 3$.

- Can we find a spectrum that is attained by an entanglement witness but not a decomposable entanglement witness?
- Determining whether or not $Conv\left(\sigma(\mathrm{EW}_{m,n})\right) = Conv\left(\sigma(\mathrm{DEW}_{m,n})\right)$ would settle a long-standing question about "absolutely separable" states.
- Specific cases of the above question might be more tractable. For example, does there exist an entanglement witness in $M_3 \otimes M_3$ with eigenvalues $(1, 1, 1, 1, 1, 1, -1, -1, c)$ for some $c < -1$?

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(preprint coming to the arXiv soon)