Completely positive completely positive maps

(and a resource theory for non-negativity of quantum amplitudes)

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Western Canada Linear Algebra Meeting 2021
Brandon, Manitoba (online)

May 30, 2021

A Tale of Two CPs: It was the best of times...

Two different usages of the phrase "completely positive" are in common usage. The first one:

- A matrix $A \in M_n(\mathbb{R})$ is called **completely positive (CP)** if there exists an entrywise non-negative matrix $B \in M_{n,m}(\mathbb{R})$ such that $A = BB^\top$. Minimal value of m is called the **CP-rank** of A.
- If A is CP then it must be positive semidefinite and entrywise non-negative (i.e., **doubly non-negative**).
- Converse is true if $n \leq 4$: easy to determine CP-ness here.
- Converse is false if $n \geq 5$: $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}$.

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- A linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is called **positive** if $\Phi(X)$ is positive semidefinite whenever X is.
- Φ is called **completely positive (CP)** if $\mathbb{1}_k \otimes \Phi$ is positive for all $k \geq 1$.
- Determining positivity seems to be hard.
- Determining *complete* positivity is easy: Φ is CP if and only if the matrix $C_\Phi := (\mathbb{1}_n \otimes \Phi)\left(\sum_{i,j=1}^n E_{i,j} \otimes E_{i,j}\right)$ is positive semidefinite.
- The matrix C_Φ is called the **Choi matrix** of Φ.

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This overloaded terminology has not caused any problems, for at least two reasons:

- The two different notions of complete positivity apply to two different types of objects: matrices, and linear maps acting on matrices.
- The two different notions of complete positivity are useful in very different situations: CP-ness of matrices comes up in convex optimization, CP-ness of linear maps comes up in operator theory and quantum information.

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Quantum Basics

Let's quickly review some of the basics of quantum information theory.

- A **quantum state** is a matrix $\rho \in M_n(\mathbb{C})$ that (a) is positive semidefinite and (b) has $\mathrm{Tr}(\rho) = 1$.
- If $\mathrm{rank}(\rho) = 1$ then is a **pure quantum state**. In this case, we can write $\rho = |v\rangle\langle v|$ for some unit vector $|v\rangle \in \mathbb{C}^n$, and we sometimes refer to $|v\rangle$ itself as a pure state.
- A **quantum channel** is a linear map $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ that (a) is completely positive and (b) is trace-preserving (i.e., $\mathrm{Tr}(\Phi(X)) = \mathrm{Tr}(X)$ for all $X \in M_n(\mathbb{C})$).

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Quantum Resource Theories: Free States

In quantum information theory, we like to classify states as "useless" or "useful" for certain tasks. This leads to the notion of a **resource theory**, which consists of two things.

1) A set $\mathcal{F}$ of **free states**:

- We typically think of these as "useless", or at least no better than classical states.
- A well-known example is the set of separable quantum states, which cannot be used for quantum teleportation, superdense coding, etc. By contrast, states that are entangled (i.e., not in $\mathcal{F}$) are "useful".
- Less-known examples include incoherent (i.e., diagonal) states, and stabilizer states.

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Quantum Resource Theories: Free Operations

2) A set of channels that send $\mathcal{F}$ to $\mathcal{F}$, called **free operations**:

- These channels cannot generate the resource—if a state is free, then applying a free operation to it keeps it free.
- For example, in the resource theory of separability/entanglement, local operations and classical communication (LOCC) channels are free, since they send separable states to separable states.

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A Resource Theory of Non-Negativity of Amplitudes

For quantum reasons that we won't get into, it is sometimes easier to work with pure states $|v\rangle$ whose amplitudes/entries are all non-negative real numbers.

- Fact: mixtures of such pure states are exactly the quantum states that are CP.
- So what can we say about the resource theory where the set of free states $\mathcal{F}$ is the set of states that are CP?

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A **purification** of a quantum state $\rho \in M_n(\mathbb{C})$ is a pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ such that $\mathrm{Tr}_1(|v\rangle\langle v|) = \rho$, where Tr_1 is the **partial trace** defined by $\mathrm{Tr}_1(A \otimes B) = \mathrm{Tr}(A)B$.

- Fact: A quantum state ρ is free (i.e., CP) if and only if it has an entrywise non-negative purification.
- Careful—if $n \geq 5$ then ρ being entrywise non-negative is not enough for it to have an entrywise non-negative purification:

$$\rho = \frac{1}{9}\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

- The minimal dimension m equals the CP-rank of ρ.

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$$\rho = \frac{1}{9}\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

- The minimal dimension m equals the CP-rank of ρ.

Purifications

A **purification** of a quantum state $\rho \in M_n(\mathbb{C})$ is a pure state $|v\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ such that $\mathrm{Tr}_1(|v\rangle\langle v|) = \rho$, where Tr_1 is the **partial trace** defined by $\mathrm{Tr}_1(A \otimes B) = \mathrm{Tr}(A)B$.

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CP-Preserving Maps

What are the free operations in this resource theory?

We say that a linear map Φ is **CP-preserving** if $\Phi(X)$ is CP whenever X is CP.

It seems like it is probably difficult to determine whether or not a given channel is CP-preserving. However, we can at least give an answer in the 2-dimensional (i.e., qubit) case, by representing quantum states in the basis of Pauli matrices:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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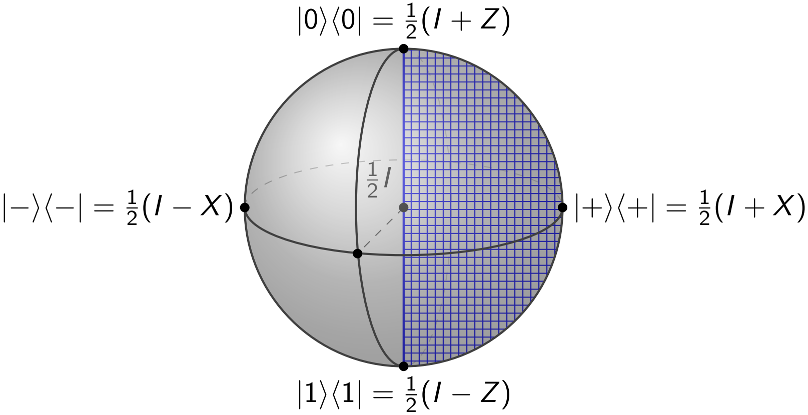

CP-Preserving Qubit Maps

Theorem (CP-Preserving Qubit Maps)

Suppose $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ *is a quantum channel with matrix representation* $[\Phi]$ *in the Pauli basis* $\{I, X, Y, Z\}$. *Then* Φ *is CP-preserving if and only if* $[\Phi]$ *has the form*

$$[\Phi] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t_x & T_{x,x} & T_{x,y} & T_{x,z} \\ 0 & 0 & T_{y,y} & 0 \\ t_z & T_{z,x} & T_{z,y} & T_{z,z} \end{bmatrix},$$

where $t_x \geq |T_{x,z}|$ *and* $T_{x,x} \geq -\sqrt{t_x^2 - T_{x,z}^2}$.

Maximally Non-CP States

In many resource theories, there is a notion of a "most resourceful" states.

In the resource theory of separability/entanglement, the "maximally entangled" states $|v\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ are those with n equal Schmidt coefficients (e.g., $(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$ if $n = 2$).

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For qubits, there are exactly two maximally non-CP states:

Theorem (Maximally Non-CP States)

There are exactly two quantum states $\sigma \in M_2(\mathbb{C})$ with the following property: for all $\rho \in M_2(\mathbb{C})$, there exists a CP-preserving channel Φ such that $\Phi(\sigma) = \rho$. They are

$$\sigma = \frac{I \pm Y}{2} = \frac{1}{2}\begin{bmatrix} 1 & \mp i \\ \pm i & 1 \end{bmatrix}.$$

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Completely Positive Completely Positive Maps

What if we ask that these free operations (i.e., CP-preserving channels) remain free upon tensoring with an identity channel (i.e., when they act on just part of a quantum state rather than the whole state)?

We say that a linear map Φ is **completely positive completely positive (CPCP)** if $\mathbb{1}_k \otimes \Phi$ is CP-preserving for all $k \geq 1$.

CPCP maps are much more straightforward to characterize than CP-preserving ones...

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Theorem (Characterization of CPCP Maps)

Suppose $\Phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is a linear map. The following are equivalent:

(a) *Φ is CPCP.*

(b) *C_Φ is completely positive.*

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Furthermore, the CP-rank of C_Φ equals the minimal number of A_j matrices possible in (c).

The A_j matrices in part (c) are called **Kraus operators** of Φ. Every CP map has Kraus operators—what makes CPCP maps special is that they have *entrywise non-negative* Kraus operators.

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Quantifying Non-CP-ness

In most resource theories, we also want ways of quantifying "how resourceful" a quantum state is.

For example, the **robustness of entanglement** is the quantity

$$R_{\mathrm{E}}(\rho) = \min_{\sigma} \left\{ s \geq 0 \ : \ \frac{\rho + s\sigma}{1+s} \text{ is separable} \right\}.$$

This roughly measures how much noise must be added to a state to make it separable (i.e., useless).

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We can do something similar in our resource theory. We define the **robustness of non-negativity** to be the quantity

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Open Questions

Too many to list them all. Some of my favourites...

- Can we compute $R_{\mathrm{NN}}(|v\rangle\langle v|)$ via semidefinite programming when $n \geq 5$?
- If $n = 2$ then the maximally non-CP states are $(I \pm Y)/2$. What are they if $n \geq 3$?
- In the resource theory of separability/entanglement, PPT states are "bound"—even though they may be entangled, no pure entanglement can be distilled from them. Is there an analogous sense in which DNN-but-not-CP states are "bound"?

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